Total Absolute Curvature and Rigidity of Surfaces in Cartan–Hadamard Manifolds

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joint work with

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AMS Fall Southeastern Sectional Meeting
Kennesaw State University
October 11, 2026

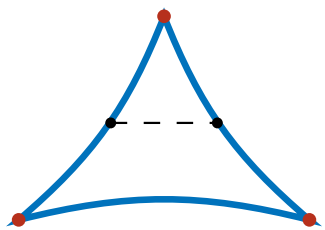
Cartan–Hadamard manifolds

M^n is Cartan–Hadamard if it is complete, simply connected, with sectional curvature $K \leq 0$.

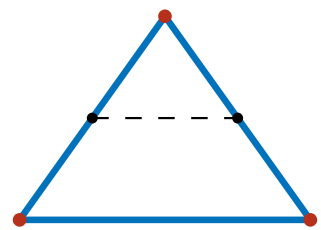
- Examples: $\mathbf{R}^n$, hyperbolic space $\mathbf{H}^n$, products, symmetric spaces of noncompact type, and vast perturbations of these.
- E.g., the space of symmetric positive definite $n \times n$ matrices, $\mathrm{GL}(n)/\mathrm{O}(n)$, with the trace metric $\langle X, Y \rangle_P := \mathrm{tr}\big(P^{-1} X P^{-1} Y\big)$, is Cartan–Hadamard.

Basic properties

- $\exp_p : T_pM \to M$ is a diffeomorphism; so M^n is diffeomorphic to $\mathbf{R}^n$, and each pair of points is joined by a unique geodesic.
- M is a CAT(0) space: a geodesic metric space where triangles are thinner than their comparison triangles in $\mathbf{R}^2$.



M



$\mathbf{R}^2$

So Cartan–Hadamard manifolds generalize $\mathbf{R}^n$ and $\mathbf{H}^n$ at the expense of losing linearity and homogeneity.

Convexity and curvature in M

A set in M is convex if it contains the geodesic connecting each pair of its points. A convex body is a compact convex set with interior points; a convex surface is the boundary of a convex body.

For a closed hypersurface Γ immersed in M^n, let GK be the determinant of the shape operator, and set

$$\mathcal{G}(\Gamma) := \int_\Gamma \mathrm{GK}, \qquad \widetilde{\mathcal{G}}(\Gamma) := \int_\Gamma |\,\mathrm{GK}\,|,$$

the total curvature and total absolute curvature of Γ.

Gauss' equation ($n = 3$). $K_\Gamma = \mathrm{GK} + K(T_p\Gamma)$,

where K_Γ is the intrinsic curvature of Γ, and $K(T_p\Gamma)$ is the sectional curvature of M on the tangent plane. Since $K \le 0$, $\mathrm{GK} \ge K_\Gamma$; so nonpositive curvature should "increase" total curvature.

Main problems

Total curvature inequality. For convex hypersurfaces $\Gamma \subset M^n$,

$$\mathcal{G}(\Gamma) \geq |\mathbf{S}^{n-1}| \,?$$

Long-standing problem (Willmore–Saleemi 1966); $|\mathbf{S}^{n-1}|$ is the volume of the unit sphere.

Cartan–Hadamard conjecture. The Euclidean isoperimetric inequality holds in M^n: $|\partial\Omega|^n \geq n^n \omega_n |\Omega|^{n-1}$ for bounded domains Ω, where ω_n is the volume of the unit ball in $\mathbf{R}^n$.

Total curvature inequality $\Rightarrow$ Cartan–Hadamard conjecture.

What is known

- The isoperimetric inequality is known for $n = 2$ (Weil 1926), $n = 4$ (Croke 1984), $n = 3$ (Kleiner 1992), and recently for $3 \leq n \leq 9$ (Chen–G.–Wang 2026).
- The total curvature inequality is known only for $n = 2$, 3, as immediate consequences of the Gauss–Bonnet theorem and Gauss' equation; for $n = 3$ it yields the isoperimetric inequality (G.–Spruck 2022).
- Both inequalities hold in $\mathbf{H}^n$, and for geodesic spheres in any Cartan–Hadamard manifold.

Recent related results

Joint work with John Stavroulakis:

- The isoperimetric inequality is locally true: perturbing the metric of the unit ball $B^n \subset \mathbf{R}^n$ to a metric with nonpositive curvature does not reduce the isoperimetric ratio (IMRN 2026).
- The total curvature inequality holds if K is constant near Γ (arXiv 2026).

The Chern–Lashof theorem

Let $\Gamma^{n-1} \subset \mathbf{R}^n$ be a closed hypersurface, immersed and oriented by the Gauss map $\nu\colon \Gamma \to \mathbf{S}^{n-1}$. Then $\mathrm{GK} = \det(d\nu)$, and

$$\widetilde{\mathcal{G}}(\Gamma) \;=\; \text{area swept out by } \nu, \text{ with multiplicity.}$$

Theorem (Chern–Lashof 1957). $\widetilde{\mathcal{G}}(\Gamma) \geq |\mathbf{S}^{n-1}|$, with equality only if Γ is the boundary of a convex body.

Proof: every height function $\langle \cdot, u \rangle$, $u \in \mathbf{S}^{n-1}$, has a max and a min on Γ; so ν covers $\mathbf{S}^{n-1}$.

Main result

Theorem (G.–Hoisington–Raffaelli–Stavroulakis). Let $\Gamma \subset M^3$ be a smooth closed immersed surface. Then

$$\widetilde{\mathcal{G}}(\Gamma) \geq 4\pi,$$

with equality only if Γ bounds a flat convex body.

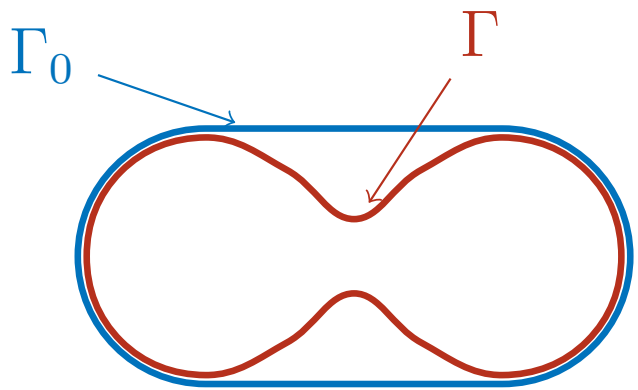
- A convex body is flat if K vanishes on it.
- Settles a problem posed by Gromov as an exercise in Ballmann–Gromov–Schroeder, *Manifolds of Nonpositive Curvature* (1985, p. 66).
- Smoothness can be relaxed to $\mathcal{C}^{1,1}$.

Some history

- Schroeder–Strake (1989): the theorem holds for strictly convex Γ.
- G.–Spruck (2023): a refinement incorporating upper bounds on the curvature of M.
- G. (2025): the theorem holds for simply connected Γ, via Alexandrov geometry, as outlined by Petrunin (2022).
- Here: the general case. The main difficulty is the equality case, due to low regularity of convex hulls, which requires a blend of metric and Riemannian techniques.

Step 0: the inequality

Let $\Gamma_0 := \partial\,\mathrm{conv}(\Gamma)$ be the boundary of the convex hull of Γ.



Observation (Kleiner 1992). The positive curvature of Γ concentrated on Γ_0 already accounts for the total curvature of Γ_0:

$$\widetilde{\mathcal{G}}(\Gamma) \geq \mathcal{G}^+(\Gamma) \geq \mathcal{G}(\Gamma \cap \Gamma_0) = \mathcal{G}(\Gamma_0),$$

where $\mathcal{G}^+(\Gamma) := \int_{\Gamma^+} \mathrm{GK}$, over the region $\Gamma^+ \subset \Gamma$ on which $\mathrm{GK} \geq 0$.

Step 0: the inequality (continued)

Γ_0 is only $\mathcal{C}^{0,1}$; so set $\mathcal{G}(\Gamma_0) := \lim_{t\to 0} \mathcal{G}(\Gamma_0^t)$ over the outer parallel surfaces Γ_0^t (points at distance t from the hull), which are $\mathcal{C}^{1,1}$.

By Gauss' equation and Gauss–Bonnet,

$$\mathcal{G}(\Gamma_0^t) = \int_{\Gamma_0^t} K_{\Gamma_0^t} - \int_{p\in\Gamma_0^t} K\big(T_p\Gamma_0^t\big) = 4\pi - \int_{p\in\Gamma_0^t} K\big(T_p\Gamma_0^t\big).$$

Since $K \le 0$,

$$\widetilde{\mathcal{G}}(\Gamma) \ge \mathcal{G}(\Gamma_0) \ge 4\pi. \qquad \blacksquare$$

So the inequality is obtained quickly. Next we prove the equality case.

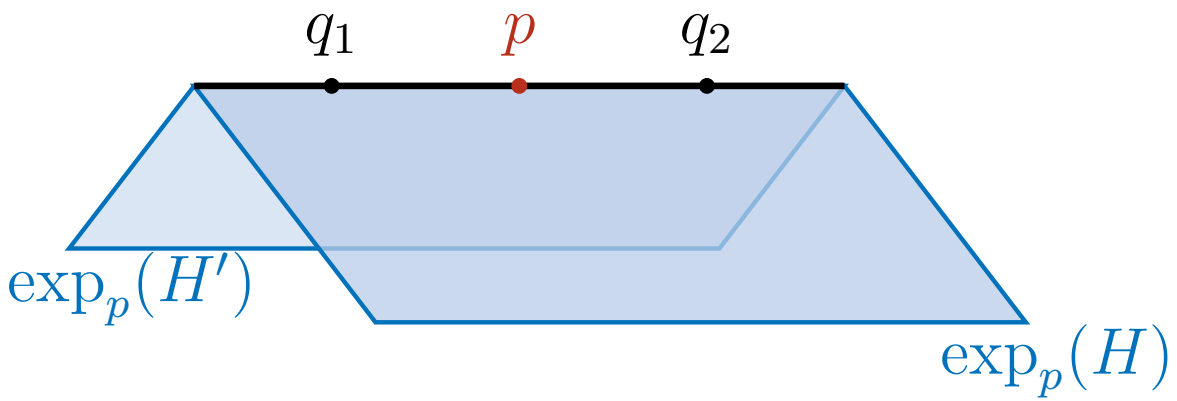
Step 1: regularity of the convex hull

Equality should force $K = 0$ on tangent planes of Γ_0 ... if Γ_0 has tangent planes! The tangent cone T_pX of $X \subset M$ at p is the limit of secant rays of $\exp_p^{-1}(X)$ emanating from p; it is flat if it is a hyperplane.

Theorem (GHRS). Let $X \subset M^3$ be compact, with $\mathrm{conv}(X)$ having interior points, and T_pX flat at each extreme point of X, i.e., each $p \in X \cap \partial\,\mathrm{conv}(X)$. Then $\mathrm{conv}(X)$ is $\mathcal{C}^1$.

Regularity: ingredients

- Locally, convex hypersurfaces in M are graphs of semiconvex functions; so flat tangent cones force $\mathcal{C}^1$ regularity (after G.–Howard 2014 in $\mathbf{R}^n$).
- At $p \in \partial C \cap X$, where $C := \mathrm{conv}(X)$: the cone T_pC is a half-space bounded by T_pX; so $T_p(\partial C) = T_pX$ is flat.
- At $p \in \partial C \setminus X$: **Lemma (Borbély 1995).** if no geodesic of M with endpoints on X passes through p, then $T_p(\partial C)$ is flat. Otherwise two support surfaces along the geodesic reach an endpoint $q_i \in X$, near which ∂C cannot be $\mathcal{C}^1$:



Step 2: flatness of tangent planes

Suppose $\widetilde{\mathcal{G}}(\Gamma) = 4\pi$. Then by Step 0, $\mathcal{G}(\Gamma_0^t) \to 4\pi$; so

$$\int_{p \in \Gamma_0^t} K\big(T_p \Gamma_0^t\big) \;\to\; 0.$$

- If $K(T_{p_0}\Gamma_0) < 0$ at some p_0 (tangent planes exist by Step 1), it stays $\le -\delta$, for some $\delta > 0$, on neighborhoods $U_t \subset \Gamma_0^t$ of definite area (the nearest-point projection is nonexpansive) — contradiction.
- So $K = 0$ on tangent planes of Γ_0.

Step 3: the gap theorem

It remains to show:

Theorem (GHRS). Let $\Gamma \subset M^3$ be a $\mathcal{C}^1$ convex surface, and suppose K vanishes on tangent planes of Γ. Then K vanishes on the convex body bounded by Γ.

"Gap theorems" of this type go back to Greene–Wu (1982) and Gromov (1985). This theorem is proved in three steps:

3.1 Construct a parallel frame along Γ.

3.2 Isometrically embed Γ into $\mathbf{R}^3$, preserving convexity and the total curvature of all $\mathcal{C}^1$ curves in Γ.

3.3 Extend the embedding to the convex body bounded by Γ.

Step 3.1: the parallel frame

Since $K \leq 0$ in M and $K = 0$ on tangent planes of Γ:

Lemma (G. 2025). $R(X, Y)Z = 0$ for all $X, Y \in T_p\Gamma$ and $Z \in T_pM$, where R is the curvature operator of M.

Lemma (GHRS). Parallel transport of TM along $\mathcal{C}^1$ curves in Γ is path-independent.

- For smooth Γ: immediate from the Ambrose–Singer holonomy theorem.
- For $\mathcal{C}^1$ surfaces: approximate homotopies between curves by smooth ones; the curvature bound makes the transport difference vanish in the limit. So Ambrose–Singer holds in the $\mathcal{C}^1$ category.

Step 3.2: the isometric embedding

For a $\mathcal{C}^1$ curve γ in M, the total curvature $\tau(\gamma)$ is the length of its tangent indicatrix: the curve traced in the unit sphere of $T_{\gamma(0)}M$ by parallel transport of the tangents γ'.

Proposition (GHRS). Let $\Gamma \subset M^3$ be a $\mathcal{C}^1$ convex surface, and suppose K vanishes on tangent planes of Γ. Then there exists a $\mathcal{C}^1$ convex isometric embedding $f\colon \Gamma \to \mathbf{R}^3$ which preserves the total curvature of all $\mathcal{C}^1$ curves in Γ.

- For $\mathcal{C}^3$ surfaces: Gauss–Codazzi equations and the fundamental theorem of surfaces (G. 2025).
- For $\mathcal{C}^1$ surfaces there is no second fundamental form; we argue instead via holonomy.

Construction of the embedding

Path-independence yields a parallel orthonormal frame e_1, e_2, e_3 of TM along Γ, with dual coframe $\theta_i := \langle \cdot, e_i \rangle$. Fix $p_0 \in \Gamma$ and set

$$f_i(p) := \int_\gamma \theta_i,$$

over any $\mathcal{C}^1$ curve γ in Γ from p_0 to p.

- $d\theta_i = 0$; so $f = (f_1, f_2, f_3)$ is well defined by Stokes' theorem.
- $df_i = \theta_i$; so f is a $\mathcal{C}^1$ isometric immersion.
- f maps the tangent indicatrix of each curve to that of its image; so it preserves total curvature of $\mathcal{C}^1$ curves.

Why convexity of the image is delicate

The image $f(\Gamma) \subset \mathbf{R}^3$ inherits nonnegative intrinsic curvature . . . but that is not enough:

Nash–Kuiper (1954–55). $\mathcal{C}^1$ isometric immersions are wildly flexible; there is no link between the intrinsic and extrinsic geometry of $\mathcal{C}^1$ surfaces in $\mathbf{R}^3$.

So we must extract extrinsic information from the construction of f. The right framework is Pogorelov's theory of surfaces with bounded extrinsic curvature.

Intrinsic curvature is nonnegative

The intrinsic curvature measure ω of Alexandrov–Zalgaller is defined via excess angles of geodesic triangles in Γ.

- **Lemma.** A $\mathcal{C}^1$ convex hypersurface in M^n is a $\mathcal{C}^1$ limit of smooth convex hypersurfaces (Greene–Wu convolution of the distance function).
- Gauss' equation for the approximations: $K_{\Gamma_i} \geq K(T\Gamma_i) \geq -\varepsilon_i$, where $\varepsilon_i := \sup |K(T\Gamma_i)| \to 0$.
- Stability of curvature bounds under Gromov–Hausdorff convergence $\Rightarrow \Gamma$ is an Alexandrov space of curvature ≥ 0, i.e., $\omega \geq 0$.

Pogorelov's theory

$\Gamma' := f(\Gamma)$ has bounded extrinsic curvature: its Gauss map has bounded variation, since convexity of Γ makes the normal a BV function in semiconvex graph coordinates.

- **Generalized Theorema Egregium (Pogorelov 1973).** For $\mathcal{C}^1$ surfaces of bounded extrinsic curvature, the extrinsic curvature measure equals the intrinsic one: $\sigma = \omega$.
- **Pogorelov.** A closed $\mathcal{C}^1$ surface in $\mathbf{R}^3$ with bounded extrinsic curvature and $\sigma \geq 0$ is convex.

Hence Γ' is convex, and f is an embedding. ■

Step 3.3: the Schur comparison theorem

Bow lemma (Schur 1921). Straightening a convex arc moves its endpoints apart.



same length, $\kappa_2 \leq \kappa_1$ $\Rightarrow$ longer chord

Extensions: hyperbolic space (Epstein 1985); polygonal "arm lemma" of Cauchy in CAT(0) spaces (Alexander–Kapovitch–Petrunin).

Schur's theorem in Cartan–Hadamard manifolds

A curve in $\mathbf{R}^2$ is chord-convex if connecting its endpoints yields a simple closed curve bounding a convex body; $|pq|$ denotes the length of the geodesic connecting p, q.

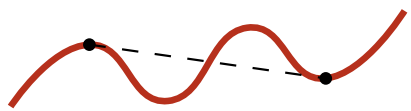
Theorem (GHRS). Let $\gamma_1 : [0, \ell] \to \mathbf{R}^2$ be chord-convex, and $\gamma_2 : [0, \ell] \to M^n$ be $\mathcal{C}^1$, with $\tau\big(\gamma_2(I)\big) \leq \tau\big(\gamma_1(I)\big)$ for every interval $I \subset [0, \ell]$. Then

$$|\gamma_2(0)\,\gamma_2(\ell)| \;\geq\; |\gamma_1(0)\,\gamma_1(\ell)|.$$

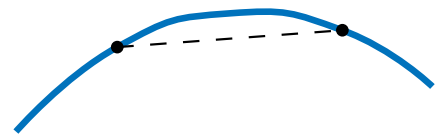
The $\mathcal{C}^2$ case was treated in G. (2025) via polygonal approximation; here we take a more analytic approach, outlined in the appendix.

Majorization

$\widetilde{\gamma}\colon [0,\ell] \to \mathbf{R}^2$ majorizes $\gamma\colon [0,\ell] \to M$ if $|\widetilde{\gamma}(t)\widetilde{\gamma}(s)| \geq |\gamma(t)\gamma(s)|$ for all t, s; properly if equality holds at the endpoints.



$\gamma \subset M$

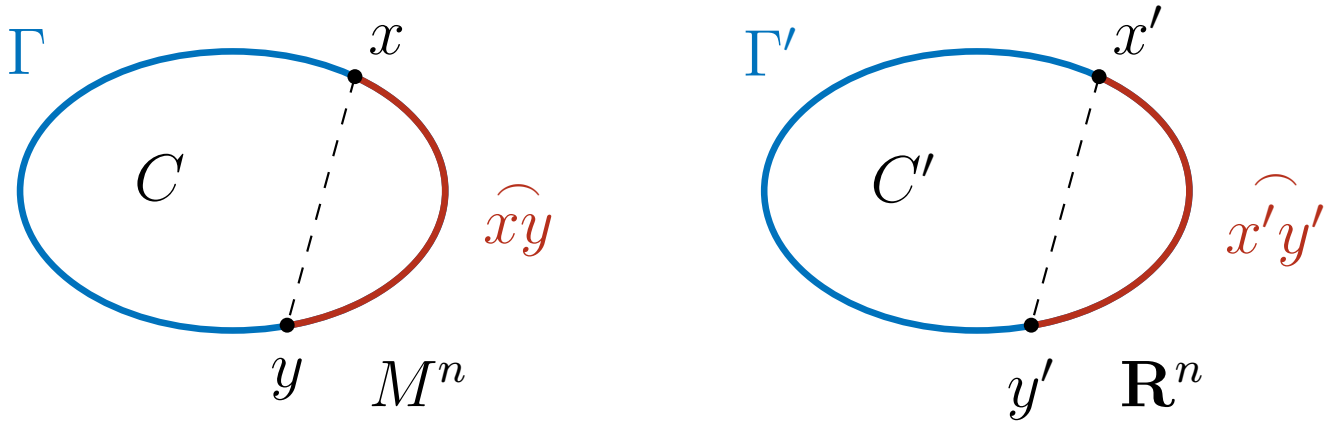


$\widetilde{\gamma} \subset \mathbf{R}^2$

Theorem (Reshetnyak 1968). Every closed curve in a CAT(0) space is properly majorized by a convex planar curve.

Rigidity of convex bodies

Proposition (GHRS). Let $C \subset M^n$ and $C' \subset \mathbf{R}^n$ be convex bodies with $\mathcal{C}^1$ boundaries Γ, Γ', and $f : \Gamma \to \Gamma'$ be a $\mathcal{C}^1$ isometry preserving the total curvature of $\mathcal{C}^1$ curves. Then f extends to an isometry $C \to C'$.



Proof of rigidity

For $x \in \Gamma$, let $x' := f(x)$. It suffices to show that $|xy| = |x'y'|$.

- Slice Γ' by planes: γ' convex; the Schur theorem applied to $\gamma = f^{-1}(\gamma')$ gives $|xy| \geq |x'y'|$.
- Conversely, Reshetnyak majorization of γ must majorize γ'; but a chord-stretching map of a convex loop is an isometry (Strantzen–Brooks 1992); so $|xy| \leq |x'y'|$.
- Extend via the generalized Kirszbraun theorem (Lang–Schroeder 1997).

Conclusion of the proof

Suppose $\widetilde{\mathcal{G}}(\Gamma) = 4\pi$.

- By Steps 1 and 2, Γ_0 is $\mathcal{C}^1$ and $K = 0$ on its tangent planes.
- By Step 3, $\mathrm{conv}(\Gamma)$ is flat, hence isometric to a convex body in $\mathbf{R}^3$, by an isometry preserving the total curvature of curves.
- Chern–Lashof now applies to the image of Γ: it is convex. So $\Gamma = \Gamma_0$ bounds a flat convex body. ■

Total positive curvature and tightness

The proof gives more: the total positive curvature satisfies

$$\mathcal{G}^+(\Gamma) \geq 4\pi,$$

with the same equality case. Surfaces with $\mathcal{G}^+(\Gamma) = 4\pi$ are called tight.

- In $\mathbf{R}^3$: $\widetilde{\mathcal{G}}(\Gamma) \geq 2\pi(2+2g)$ for genus g, with equality exactly for tight surfaces.
- Solanes (2007): in $\mathbf{H}^3$ there are examples with $\widetilde{\mathcal{G}}(\Gamma) < 2\pi(2+2g)$ for every $g \geq 1$.
- So minimizers of $\widetilde{\mathcal{G}}$ and $\mathcal{G}^+$ diverge in Cartan–Hadamard manifolds, and $\mathcal{G}^+$ is the right notion of tightness: all tight surfaces in M^3 lie in flat convex bodies.

Open problems

- Problem of S. Alexander: is a closed surface with $\mathrm{GK} \geq 0$ in M^3 convex? True in $\mathbf{R}^3$ by Chern–Lashof (Hadamard 1898 for $\mathrm{GK} > 0$), and in M^3 for $\mathrm{GK} > 0$ (Alexander 1977).
- Regularity of convex hulls in M^n, $n \geq 4$: does flatness of tangent cones still imply $\mathcal{C}^1$ regularity? (Borbély's lemma fails as stated.)
- A Pogorelov theory for hypersurfaces of bounded extrinsic curvature in higher dimensions?
- Does *every* majorization of a $\mathcal{C}^1$ curve fail to increase its curvature?

Thanks for your attention!

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Appendix: proof of the Schur theorem

Key Proposition (GHRS). Every $\mathcal{C}^1$ curve in M admits a chord-convex $\mathcal{C}^1$ proper majorization which does not increase curvature. Then the classical Schur theorem in $\mathbf{R}^2$ completes the proof.

- For $\mathcal{C}^{1,1}$ curves this follows from curvature–chord estimates via expansions in normal coordinates (next slides).
- The $\mathcal{C}^1$ case follows by approximation.

Appendix: curvature–chord estimates

In normal coordinates centered at $o \in M$, for $a, b \in T_oM$,

$$\left|\exp_o(a)\exp_o(b)\right|^2 = |ab|^2 - \tfrac{1}{3}R(a,b,b,a) + O\big((|a|+|b|)^5\big).$$

Consequently:

- At twice differentiable points of a $\mathcal{C}^1$ unit speed curve γ,
$$|\gamma(t-h)\,\gamma(t+h)| = 2h - \tfrac{\kappa(t)^2}{3}\,h^3 + o(h^3).$$
- For $\mathcal{C}^{1,1}$ curves, uniformly in t:
$|\gamma(t-h)\,\gamma(t+h)| \geq 2h - Ch^3$ (here $K \leq 0$ enters: $\exp_o$ is noncontracting).

Appendix: the $\mathcal{C}^{1,1}$ case

Let γ be $\mathcal{C}^{1,1}$, and $\widetilde{\gamma}$ be *any* chord-convex majorization of γ.

- Majorization and the uniform estimate give $|D_h^2\widetilde{\gamma}(t)| \leq Ch^2$ for the second symmetric differences; hence $\widetilde{\gamma}$ is $\mathcal{C}^{1,1}$.
- Comparing the chord expansions of $\widetilde{\gamma}$ and γ at twice differentiable points,

$$2h - \tfrac{\widetilde{\kappa}^2}{3}h^3 + o(h^3) \;\geq\; 2h - \tfrac{\kappa^2}{3}h^3 + o(h^3),$$

  yields $\widetilde{\kappa} \leq \kappa$ almost everywhere.

So majorization does not increase the curvature of $\mathcal{C}^{1,1}$ curves.

Appendix: the $\mathcal{C}^1$ case

- Approximate γ by $\mathcal{C}^{1,1}$ curves γ_i (geodesic polygons with rounded corners), with $\tau(\gamma_i(I)) \to \tau(\gamma(I))$.
- Majorize each γ_i by a chord-convex $\mathcal{C}^{1,1}$ curve $\widetilde{\gamma}_i$ with nondecreasing turning angle $\theta_i \colon [0, \ell] \to [0, 2\pi]$, i.e., $\widetilde{\gamma}_i' = (\cos\theta_i, \sin\theta_i)$.
- **Helly's selection theorem.** $\theta_i \to \theta$ pointwise, θ nondecreasing. Define $\widetilde{\gamma}$ by

$$\widetilde{\gamma}(0) = (0,0), \qquad \widetilde{\gamma}'(t) = \big(\cos\theta(t),\, \sin\theta(t)\big).$$

- Then $\widetilde{\gamma}$ is convex, $\mathcal{C}^1$, majorizes γ, and $\tau(\widetilde{\gamma}(I)) = \lim \tau(\widetilde{\gamma}_i(I)) \le \tau(\gamma(I))$. ■

Appendix: the simply connected case

How was the simply connected case proved in G. (2025)?
Suppose $\Gamma \subset M^3$ is simply connected and $\widetilde{\mathcal{G}}(\Gamma) = 4\pi$.

- Gauss' equation: $\mathrm{GK} = K_\Gamma - K(T_p\Gamma) \geq K_\Gamma$.
- Gauss–Bonnet: $\mathcal{G}(\Gamma) = \int_\Gamma K_\Gamma = 4\pi$.
- So $\widetilde{\mathcal{G}}(\Gamma) \geq \mathcal{G}(\Gamma) \geq 4\pi$; equality forces $\mathrm{GK} \geq 0$ and $K(T_p\Gamma) = 0$ on Γ.
- Since Γ is smooth in that setting, the isometric embedding may be constructed via the Gauss–Codazzi equations, and Γ is convex.