Total Curvature and Isoperimetric Inequalities in Pinched Cartan–Hadamard Manifolds

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Cartan–Hadamard manifolds

M^n is Cartan–Hadamard if it is complete, simply connected, with sectional curvature $K \leq 0$.

- $\exp_p : T_pM \to M$ is a diffeomorphism; so each pair of points is joined by a unique geodesic.
- Examples: $\mathbf{R}^n$, hyperbolic space $\mathbf{H}^n$, products, symmetric spaces of noncompact type, and vast perturbations of these.
- M has pinched curvature if $-b^2 \leq K \leq -a^2 < 0$.

Total curvature of convex hypersurfaces

A set in M is convex if it contains the geodesic connecting each pair of its points. A convex hypersurface $\Gamma \subset M^n$ is the boundary of a compact convex set with interior points.

The total Gauss–Kronecker curvature of a smooth Γ is

$$\mathcal{G}(\Gamma) := \int_\Gamma \mathrm{GK}, \qquad \mathrm{GK} := \det(A),$$

where A is the shape operator of Γ.

The total curvature inequality

This talk concerns the inequality

$$\mathcal{G}(\Gamma) \geq |\mathbf{S}^{n-1}|. \qquad (*)$$

- In $\mathbf{R}^n$, equality holds in $(*)$, since the Gauss map is onto.
- **Theorem (Borbély 2002; G.–Spruck 2022).** $(*)$ holds in $\mathbf{H}^n$.
- For $n = 3$, $(*)$ holds in *every* Cartan–Hadamard manifold, via Gauss–Bonnet and the Gauss equation.

A long-standing problem

Problem (Willmore–Saleemi 1966; BGS 1985).
Does $(*)$ hold for convex hypersurfaces in all Cartan–Hadamard manifolds M^n, $n \geq 4$?

Theorem (G.–Spruck 2022). $(*)$ in M^n implies the Cartan–Hadamard conjecture in M^n: for bounded $\Omega \subset M$,

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B^n|^n}{|B^n|^{n-1}},$$

where B^n is the unit ball in $\mathbf{R}^n$.

- Isoperimetric inequality known for $n = 2$ (Weil 1926), $n = 4$ (Croke 1984), $n = 3$ (Kleiner 1992), and recently for $3 \leq n \leq 9$ (Chen–G.–Wang 2026).
- $(*)$ itself is known only for $n = 3$.

Total mean curvatures

Let $\sigma_j(A)$ be the j^{th} elementary symmetric function of the principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$ of Γ, i.e., the eigenvalues of A. For $0 \leq j \leq n-1$, the total mean curvatures of Γ are

$$\mathcal{M}_j(\Gamma) := \int_\Gamma \sigma_j(A).$$

- $\mathcal{M}_0 = |\Gamma|$, the area of Γ.
- $\mathcal{M}_1$ is the usual total mean curvature.
- $\mathcal{M}_{n-1} = \mathcal{G}(\Gamma)$, since $\sigma_{n-1}(A) = \det(A) = \text{GK}$.

Main result I: small diameter

Theorem 1 (G. 2026). For $n \geq 4$, let M^n be a Cartan–Hadamard manifold with $-b^2 \leq K \leq -a^2 < 0$. Then there exists $\delta_n > 0$, depending only on n, such that $(*)$ holds for every smooth convex hypersurface $\Gamma \subset M$ with

$$\operatorname{diam}(\Gamma) \leq \delta_n \frac{a}{b^2}.$$

- $\operatorname{diam}(\Gamma) := \sup_{x,y\in\Gamma} \operatorname{dist}_M(x,y)$.
- The proof gives a quantitative defect bound:

$$\mathcal{G}(\Gamma) - |\mathbf{S}^{n-1}| \geq \frac{a^2}{2(n-2)} \mathcal{M}_{n-3}(\Gamma).$$

Main result II: dimensions 4 and 5

$$\lambda_4 := \frac{1}{\sqrt{3}} \approx 0.58, \qquad \lambda_5 := \frac{1+\sqrt{113}}{14} \approx 0.83.$$

Theorem 2 (G. 2026). For $n = 4, 5$, let M^n be a Cartan–Hadamard manifold with $-b^2 \leq K \leq -\lambda_n b^2$, for some $b > 0$. Then $(*)$ holds for every smooth convex hypersurface $\Gamma \subset M^n$ — no restriction on the diameter.

- First instances of $(*)$ in dimensions > 3 with nonconstant negative curvature (with Theorem 1).
- The pinching is needed only near Γ.

Corollary: isoperimetric inequality

Corollary. Let M^5 be a Cartan–Hadamard manifold with $-b^2 \le K \le -\lambda_5 b^2$. Then every bounded set $\Omega \subset M$ of finite perimeter and positive volume satisfies

$$|\partial\Omega| > |\partial B|,$$

where B is a ball in $\mathbf{R}^5$ with $|B| = |\Omega|$.

- Cartan–Hadamard conjecture in dimension 5 for nonconstant negative curvature; cf. Chen–G.–Wang (2026).
- Pinching does not force constant curvature: for all $0 < a < b$ there are examples with $-b^2 \le K \le -a^2$, e.g. universal covers of Gromov–Thurston (1987) manifolds.

Chern–Gauss–Bonnet

For a compact oriented N^n, n even, with smooth boundary ∂N,

$$|\mathbf{S}^{n-1}|\,\chi(N) = \frac{2|\mathbf{S}^{n-1}|}{|\mathbf{S}^{n}|}\int_N \mathrm{Pf}(R) + \int_{\partial N} \mathrm{TPf},$$

where the Pfaffian $\mathrm{Pf}(R)$ is a polynomial of degree $n/2$ in the curvature tensor R, with $\mathrm{Pf}(R_1) = 1$, and TPf is Chern's transgression form, built from the second fundamental form of ∂N and the ambient curvature.

Expansion of the transgression

Lemma (G. 2026). In a principal frame, with κ_i the principal curvatures and $K_{ij} := K(e_i, e_j)$,

$$\mathrm{TPf} = \mathrm{GK} + \frac{1}{n-2} L + E, \qquad L := \sum_{i<j} K_{ij} \prod_{\ell \neq i,j} \kappa_\ell,$$

where E collects the terms with ≥ 2 ambient curvature factors: $|E| \leq c_n \sum_{s \geq 2} |R|^s \sigma_{n-1-2s}(A)$.

Key sign: $K \leq -a^2$ and $\kappa_\ell \geq 0$ give $L \leq -a^2 \sigma_{n-3}(A)$.

Defect formulas

Let Ω be the convex body bounded by Γ.

Proposition (n even). Since $\chi(\Omega) = 1$,

$$\mathcal{G}(\Gamma) - |\mathbf{S}^{n-1}| = -\frac{2|\mathbf{S}^{n-1}|}{|\mathbf{S}^{n}|} \int_\Omega \mathrm{Pf}(R) - \frac{1}{n-2} \int_\Gamma L - \int_\Gamma E.$$

Proposition (n odd). Since $\Gamma \cong \mathbf{S}^{n-1}$, intrinsic Chern–Gauss–Bonnet and the Gauss equation give

$$\mathcal{G}(\Gamma) - |\mathbf{S}^{n-1}| = -\frac{1}{n-2} \int_\Gamma L - \int_\Gamma E.$$

The term $-\int_\Gamma L$ is favorable; it remains to control E and the Pfaffians by the total mean curvatures $\mathcal{M}_j$.

Why dimensions 4 and 5?

In dimensions 4 and 5 the defect formulas simplify dramatically:

- Each term of E contains two or more ambient curvature factors, and each factor replaces a pair of principal curvatures. For $n \leq 5$ there is no room for them, beyond a single Pfaffian.
- $n = 4$: $E = 0$, and only the interior Pfaffian integral is unfavorable.
- $n = 5$: $E = \mathrm{Pf}\left(R|_{T\Gamma}\right)$, with no interior term.
- Both Pfaffians are 4-dimensional, where self-duality shows they are maximized by constant curvature.
- The resulting volume and area terms are absorbed into $-\int_\Gamma L$ via diameter-free estimates for $\mathcal{M}_1$ and $\mathcal{M}_2$.

A sharp Pfaffian bound in dimension 4

Normalize $\mathrm{Pf}(R_k) = k^2$ for the constant curvature tensor R_k on $\mathbf{R}^4$.

Lemma (G. 2026). If R is an algebraic curvature tensor on $\mathbf{R}^4$ with $-1 \le K \le 0$, then

$$\mathrm{Pf}(R) \;\le\; 1 - \left(1 + \frac{\mathrm{Scal}_R}{12}\right)^2 \;\le\; 1,$$

with $\mathrm{Pf}(R) = 1$ if and only if $R = R_{-1}$.

- Refines a theorem of Ville (1987); see also Bettiol–Kummer–Mendes (2024).
- Proof: block decomposition of the curvature operator on $\Lambda^2\mathbf{R}^4 = \Lambda^+ \oplus \Lambda^-$, and eigenvalue estimates.

The classical Minkowski formula

Let Γ be a smooth closed hypersurface in $\mathbf{R}^n$, with position vector x and outward unit normal ν.

Minkowski formula.

$$\int_\Gamma \langle x, \nu \rangle \, \sigma_1(A) \;=\; (n-1)\,|\Gamma|.$$

- When Γ is convex and the origin lies inside it, $\langle x, \nu \rangle$ is the support function of Γ.
- Follows from the Stokes theorem applied to the tangential component of the position vector field x.

The Hsiung–Minkowski formulas

Theorem (Hsiung 1954). For a smooth closed hypersurface $\Gamma \subset \mathbf{R}^n$, and $1 \leq j \leq n-1$,

$$j \int_\Gamma \langle x, \nu \rangle \, \sigma_j(A) \;=\; (n-j) \int_\Gamma \sigma_{j-1}(A).$$

- $j = 1$ recovers the classical Minkowski formula.
- These identities relate the successive total mean curvatures $\mathcal{M}_{j-1}$, $\mathcal{M}_j$, weighted by the support function; see Montiel–Ros (1991).
- Note that $\langle x, \nu \rangle = \langle \nabla f, \nu \rangle$ for $f := |x|^2/2$. This is the form which generalizes to Cartan–Hadamard manifolds.

Weighted Hsiung–Minkowski identity

The Newton transformations of the shape operator A are

$$T_0 := I, \quad T_j := \sigma_j(A)\, I - A T_{j-1}, \quad 0 \le T_{j-1} \le \sigma_{j-1}(A)\, I.$$

Proposition (G. 2026). For smooth f near Γ, and $1 \le j \le n-1$,

$$j \int_\Gamma \langle \nabla f, \nu \rangle \sigma_j(A) = \int_\Gamma \operatorname{tr}\left(T_{j-1} \nabla^2 f|_{T\Gamma}\right) + \int_\Gamma \langle \operatorname{div} T_{j-1}, \nabla_\Gamma f \rangle.$$

- $f := |x|^2/2$ in $\mathbf{R}^n$ recovers the classical Hsiung–Minkowski formulas, where $\operatorname{div} T_{j-1} = 0$.

Divergence estimates

The divergence term is absent from the classical formulas; in M it is controlled by the ambient curvature:

- $\operatorname{div} T_0 = 0$.
- The Codazzi equation gives $|\operatorname{div} T_{j-1}| \leq c_n |R|\, \sigma_{j-2}(A)$.
- $\operatorname{div} T_1 = -\big(\operatorname{Ric}(\nu)\big)^\top$, so Berger's lemma gives

$$|\operatorname{div} T_1| \leq \frac{n-2}{2}(1-\lambda)$$

when $-1 \leq K \leq -\lambda$.

Small diameter: interior radial estimates

Fix p in Ω, and set $f := r^2/2$, where $r := \mathrm{dist}_M(p, \cdot)$.

- Hessian comparison: $\nabla^2 f \geq g$; also $|\nabla f| = r \leq d := \mathrm{diam}(\Gamma)$.
- The weighted identity gives, when the curvature tensor of M satisfies $|R| \leq B^2$ on Ω for some constant $B > 0$,

$$(n-j)\,\mathcal{M}_{j-1} \leq \left(j\mathcal{M}_j + c_n B^2 \mathcal{M}_{j-2}\right) d.$$

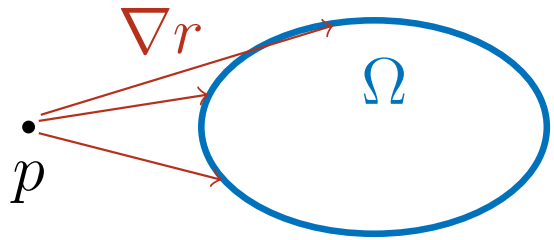
- Iterating these bounds, for $Bd \leq \varepsilon_n$, controls $\mathcal{M}_{n-1-2s}$ and $|\Omega|$ by $\mathcal{M}_{n-3}$.
- Feeding these into the defect formula:

$$\mathcal{G}(\Gamma) - |\mathbf{S}^{n-1}| \geq \left(\frac{a^2}{n-2} - c_n B^4 d^2\right)\mathcal{M}_{n-3}(\Gamma),$$

which yields Theorem 1.

Diameter-free: exterior radial estimates

Fix $p \notin \Omega$, and set $r := \mathrm{dist}_M(p, \cdot)$, which is smooth on Ω.



- Hessian comparison for $K \le -a^2$: $\nabla^2 r(X, X) \ge a\big(|X|^2 - \langle \nabla r, X\rangle^2\big)$.
- Weighted identity, Cauchy–Schwarz, and factoring yield:

Proposition (G. 2026). $\mathcal{M}_1 \ge (n-1)^2 a^2 \,|\Omega|$; and if $-1 \le K \le -\lambda$, $n \ge 4$,

$$2\mathcal{M}_2 \;\ge\; \Big((n-2)^2\lambda - \frac{n-2}{2}(1-\lambda)\Big)|\Gamma|.$$

The case $j = 1$ also recovers Yau's linear isoperimetric inequality $|\Gamma| \ge (n-1)a|\Omega|$.

Proof in dimension 4

After scaling, $-1 \leq K \leq -\lambda$, $\lambda \geq \lambda_4$.

- Defect formula ($E = 0$, and $2|\mathbf{S}^3|/|\mathbf{S}^4| = 3/2$):

$$\mathcal{G}(\Gamma) - |\mathbf{S}^3| = -\frac{3}{2}\int_\Omega \mathrm{Pf}(R) - \frac{1}{2}\int_\Gamma L.$$

- $L \leq -\lambda\,\sigma_1(A)$, so $-\frac{1}{2}\int_\Gamma L \geq \frac{\lambda}{2}\mathcal{M}_1$.
- Pfaffian bound and $\mathcal{M}_1 \geq 9\lambda|\Omega|$:

$$\int_\Omega \mathrm{Pf}(R) \;\leq\; |\Omega| \;\leq\; \frac{\mathcal{M}_1}{9\lambda}.$$

- Hence

$$\mathcal{G}(\Gamma) - |\mathbf{S}^3| \;\geq\; \Big(\frac{\lambda}{2} - \frac{1}{6\lambda}\Big)\mathcal{M}_1 = \frac{3\lambda^2 - 1}{6\lambda}\,\mathcal{M}_1 \;\geq\; 0. \quad \blacksquare$$

Proof in dimension 5

After scaling, $-1 \leq K \leq -\lambda$, $\lambda \geq \lambda_5$.

- $\Gamma \cong \mathbf{S}^4$; the intrinsic defect formula:

$$\mathcal{G}(\Gamma) - |\mathbf{S}^4| = -\frac{1}{3}\int_\Gamma L - \int_\Gamma \mathrm{Pf}\left(R|_{T\Gamma}\right).$$

- $L \leq -\lambda\,\sigma_2(A)$ and $\mathrm{Pf}\left(R|_{T\Gamma}\right) \leq 1$ give

$$\mathcal{G}(\Gamma) - |\mathbf{S}^4| \;\geq\; \frac{\lambda}{3}\mathcal{M}_2 - |\Gamma|.$$

- The $\mathcal{M}_2$ estimate, with $n = 5$: $2\mathcal{M}_2 \geq \frac{21\lambda - 3}{2}\,|\Gamma|$.
- Hence

$$\mathcal{G}(\Gamma) - |\mathbf{S}^4| \;\geq\; \frac{7\lambda^2 - \lambda - 4}{4}\,|\Gamma| \;\geq\; 0. \quad \blacksquare$$

An obstruction in dimension 6

- In dimension 6 the method meets an obstruction: Geroch (1976) constructed a curvature tensor with $K > 0$ and $\mathrm{Pf} < 0$; replacing R by $-R$ shows that $K \leq 0$ does *not* determine the sign of the Pfaffian.
- Moreover $\mathcal{M}_1(\Gamma)/|\Omega|$ is unbounded (small geodesic balls, thin tubes), so the remaining error terms cannot be absorbed.

Open problems

- Does $(*)$ hold under pinching for $n \geq 6$?
- Improve λ_4, λ_5? Both proofs discard information.
- Total curvature conjecture: $(*)$ for convex hypersurfaces in all Cartan–Hadamard manifolds M^n $\Rightarrow$ Cartan–Hadamard conjecture.

Thanks for your attention!

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Total curvature and isoperimetric inequalities in pinched Cartan–Hadamard manifolds

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