

Isoperimetric Inequality in Spaces of Nonpositive Curvature



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The classical isoperimetric inequality

For any bounded set $\Omega \subset \mathbf{R}^{n \geq 2}$, with boundary Γ ,

$$\frac{|\Gamma|^n}{|\Omega|^{n-1}} \geq \frac{|\mathbf{S}^{n-1}|^n}{|B^n|^{n-1}},$$

where B^n is the unit ball in $\mathbf{R}^n$, and $\mathbf{S}^{n-1}$ is the unit sphere.



In this talk we discuss the generalization of this inequality to *Cartan-Hadamard manifolds*, spaces of nonpositive curvature which generalize $\mathbf{R}^n$.

Applications

Applications: Sobolev Inequality (with sharp constant)

For every smooth function f with compact support in $\mathbf{R}^n$,

$$\left(\int_{\mathbf{R}^n} |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq \frac{1}{n|B^n|^{\frac{1}{n}}} \int_{\mathbf{R}^n} |\nabla f|.$$

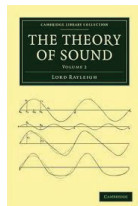
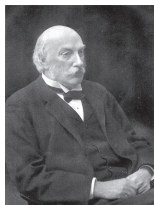
Letting f approach the characteristic function of Ω gives back the isoperimetric inequality.

Applications: Faber-Krahn Inequality

Let λ_1 denote the first Dirichlet eigenvalue. Then,

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

where B is a ball with $|B| = |\Omega|$; furthermore, equality holds only if Ω is a ball.



The inequality was proved in the 1920s, as had been conjectured by Lord Rayleigh in his book, *Theory of Sound*, in 1877.

Sobolev and Faber-Krahn inequalities both hold on Cartan-Hadamard manifolds provided that the isoperimetric inequality holds there as well. Many other inequalities would follow too.

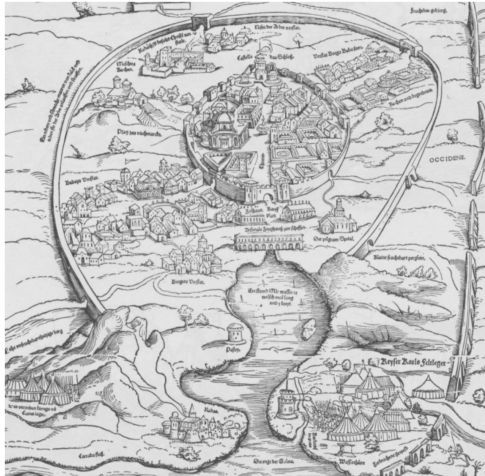
Early History and Classical Proofs of the isoperimetric inequality

Queen Dido of Carthage, 9th century B.C.



According to Virgil (*Aeneid*, 29–19 BC) Dido was allowed to claim as much land for Carthage as could be enclosed within an Ox's hide (cut into thin ribbons).

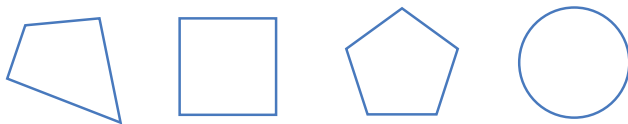
Queen Dido of Carthage, 9th century B.C.



Map of Tunis (modern Carthage) from 1535.

Zenodorus, 150 B.C.

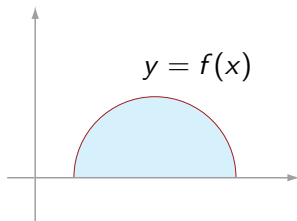
In *On isoperimetric figures* proves the isoperimetric inequality for polygonal curves in the plane.



Reported to have also studied the 3-dimensional problem, but the book is lost (likely it burned with the library of Alexandria in 48 BC).

Euler 1744

Prompted by the Bernoulli brothers, proves that the area under the graph of a smooth function is maximized by a semicircle.

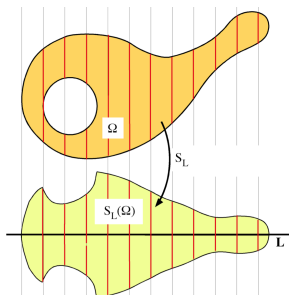


In the process discovers the Euler-Lagrange equation which lays the foundations for *Calculus of Variations*. Let $\Phi := y + \lambda\sqrt{1 + y'^2}$. Then

$$\Phi_y - \frac{d}{dx}\Phi_{y'} = 0.$$

Steiner, 1838

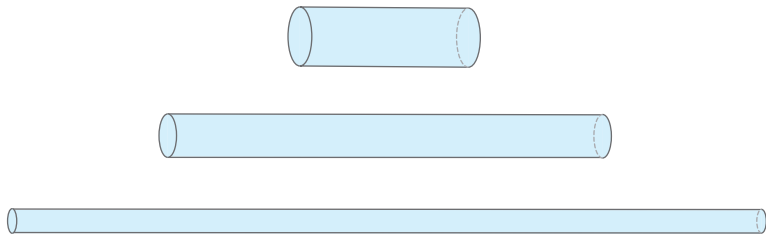
Discovers a symmetrization which preserves volume, and reduces perimeter of any compact domain $\Omega \subset \mathbf{R}^n$ with regular boundary.



So a perimeter minimizer (if it exists!) must be a sphere.

Existence of a minimizer

Existence of a minimizer is not obvious, as first pointed out and remedied by Schwarz and Weierstrass.



A sequence of domains with fixed volume and decreasing perimeter may not have any reasonable limit.

Existence of a minimizer

A minimizer exists because:

- ▶ Steiner symmetrization does not increase the outradius.
- ▶ **Blaschke Selection Principle:** The space of compact subsets of a metric space is locally compact with respect to Hausdorff distance.



Symmetrization can be used to establish the isoperimetric inequality in the hyperbolic space $\mathbf{H}^n$, but it does not work in a general Riemannian space, due to absence of any symmetry or linear structure.

We need a more local or *variational approach* ...

Brunn-Minkowski (-Lyusternik) Inequality, 1887–1935

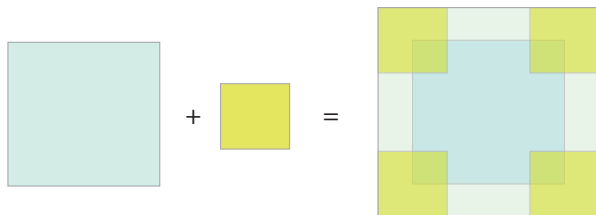


For $\Omega_1, \Omega_2 \subset \mathbf{R}^n$

$$|\Omega_1 + \Omega_2|^{\frac{1}{n}} \geq |\Omega_1|^{\frac{1}{n}} + |\Omega_2|^{\frac{1}{n}}.$$

Brunn-Minkowski (-Lyusternik) Inequality, 1887–1935

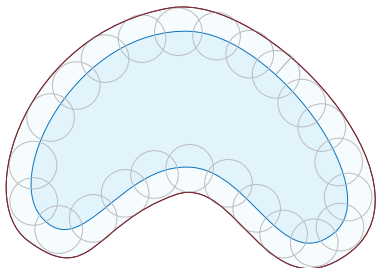
$$|\Omega_1 + \Omega_2|^{\frac{1}{n}} \geq |\Omega_1|^{\frac{1}{n}} + |\Omega_2|^{\frac{1}{n}}.$$



Proof is immediate for rectangles. Follows for all regions by approximation with rectangles and induction (takes only one page!).

Proof of Isoperimetric Inequality from Brunn-Minkowski

$$\Omega_r := \Omega + rB = \{x \in \mathbf{R}^n \mid \text{dist}(\Omega, x) \leq r\}$$



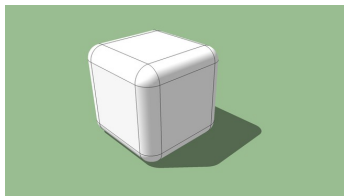
Proof of Isoperimetric Inequality from Brunn-Minkowski

$$\begin{aligned} |\Omega_r| &= |\Omega + rB| \\ &\geq \left(|\Omega|^{\frac{1}{n}} + r|B|^{\frac{1}{n}} \right)^n \\ &\geq |\Omega| + nr|\Omega|^{\frac{n-1}{n}}|B|^{\frac{1}{n}} \end{aligned}$$

$$\begin{aligned} |\Gamma| &= \lim_{r \rightarrow 0} \frac{|\Omega_r| - |\Omega|}{r} \\ &\geq n|\Omega|^{\frac{n-1}{n}}|B|^{\frac{1}{n}} \end{aligned}$$

Steiner Polynomial

If Ω is a convex polyhedron in $\mathbf{R}^3$, with edge lengths L_i and corresponding dihedral angles θ_i ,



$$|\Omega_r| = |\Omega| + |\Gamma|r + \left(\sum L_i(\pi - \theta_i) \right) r^2 + \frac{4}{3}\pi r^3.$$

If Γ is smooth,

$$|\Omega_r| = |\Omega| + |\Gamma|r + \left(\frac{1}{2} \int_{\Gamma} H \right) r^2 + \left(\frac{1}{3} \int_{\Gamma} GK \right) r^3.$$

Generalized mean curvatures

If Γ is a level set of a function f , then, at twice differentiable points, the *principal curvatures* of Γ

$$\kappa := (\kappa_1, \dots, \kappa_{n-1}),$$

are given by the eigenvalues of the Hessian of f restricted to the tangent hyperplanes of Γ , divided by the norm of the gradient of f . We denote the elementary symmetric functions of these curvatures by

$$\sigma_\ell(\kappa) := \sum_{i_1 < \dots < i_\ell} \kappa_{i_1} \dots \kappa_{i_\ell},$$

and call these the (unnormalized) *generalized mean curvatures* of Γ . In particular,

$$\sigma_1(\kappa) = H, \quad \text{and} \quad \sigma_{n-1}(\kappa) = GK.$$

Steiner polynomial

If $\Gamma \subset \mathbf{R}^n$ is a smooth closed embedded hypersurface with outward normal ν , then

$$\Gamma_t := \Gamma + t\nu$$

is well-defined for small t , and

$$|\Gamma_t| = |\Gamma| + C_1 \mathcal{M}_1(\Gamma) t + \cdots + C_{n-1} \mathcal{M}_{n-1}(\Gamma) t^{n-1},$$

where C_r are constants depending only on n , and

$$\mathcal{M}_r(\Gamma) := \int_{\Gamma} \sigma_r(\kappa)$$

is the r -th total mean curvature. With $\sigma_0 := 1$, we have $\mathcal{M}_0 = |\Gamma|$; $\mathcal{M}_1$ is the total (first) mean curvature, and $\mathcal{M}_{n-1}$ is the total Gauss-Kronecker curvature.

Steiner polynomial

Proof. Let $f: \Gamma \rightarrow \Gamma_t$ be given by

$$f(x) := x + t\nu(x).$$

Then

$$df = I + t d\nu,$$

where $d\nu$ is the *shape operator* of Γ , whose eigenvalues are the principal curvatures κ_i . So

$$|\Gamma_t| = \int_{\Gamma} \det(df) = \int_{\Gamma} \det(I + t d\nu),$$

which is the integral of the characteristic polynomial of the shape operator. □

The Cartan-Hadamard Conjecture

Cartan-Hadamard manifolds

A Cartan-Hadamard manifold is a complete simply connected manifold of nonpositive curvature.



Negatively curved surfaces occur throughout nature. They pack more area inside each geodesic disc, compared to surfaces of zero curvature, through their waves or corrugations.

Examples of Cartan-Hadamard manifolds

- ▶ $\mathbf{R}^n \times \mathbf{H}^m$, and more generally products of Cartan-Hadamard manifolds.

- ▶ The space of positive definite $n \times n$ matrices P , with the metric

$$ds^2 = \operatorname{tr}((P^{-1}dP)^2).$$

(Covariance matrices, ellipsoids, flat tori.)

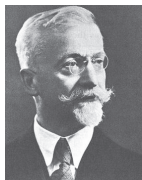
- ▶ The Siegel upper half-space of symmetric complex $g \times g$ matrices $Z = X + iY$ with $Y > 0$, and

$$ds^2 = \operatorname{tr}(Y^{-1}dZ Y^{-1}d\bar{Z}).$$

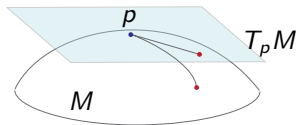
(Period matrices of Riemann surfaces.)

- ▶ The unit ball in $\mathbb{C}^n$ with its Bergman metric, whose curvature is not constant: $-4 \leq K \leq -1$.

Cartan-Hadamard manifolds



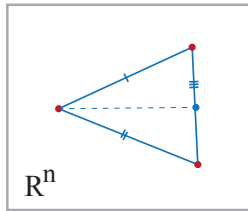
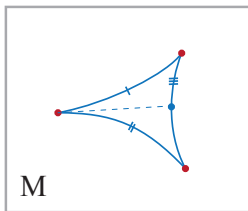
Cartan and Hadamard showed that in a Cartan-Hadamard manifold the exponential map is a diffeomorphism.



So Cartan-Hadamard manifolds are diffeomorphic to $\mathbf{R}^n$ and every pair of points may be connected with a unique geodesic.

Cartan-Hadamard manifolds

Nonpositive curvature means that triangles are “thinner” than those in $\mathbf{R}^n$ (Toponogov’s theorem).



Alternatively, it also means that geodesics emerging from a point diverge faster than those in $\mathbf{R}^n$. That rate is measured by the Riemann curvature tensor (Jacobi’s Equation).

Cartan-Hadamard manifolds



$$C = 2\pi D$$



Zero curvature

(a)



$$C < 2\pi D$$



Positive curvature

(b)



$$C > 2\pi D$$



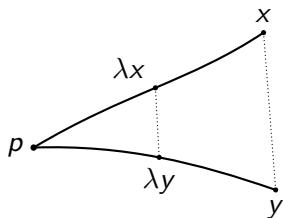
Negative curvature

(c)

Since in a Cartan-Hadamard manifold geodesics diverge faster than in $\mathbf{R}^n$, it follows from standard Riemannian comparison theory, that geodesic balls satisfy the isoperimetric inequality.

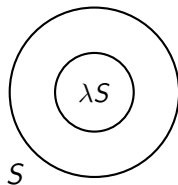
Cartan-Hadamard manifolds

Another way to think of nonpositive curvature: the exponential map is *expansive*. For $x \in M$ and $0 \leq \lambda \leq 1$, let $\lambda x := \exp_p(\lambda \exp_p^{-1}(x))$.



$$d(\lambda x, \lambda y) \leq \lambda d(x, y)$$

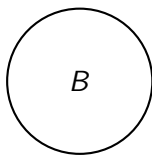
So, if S is a geodesic sphere centered at p ,



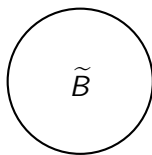
$$|\lambda S| \leq \lambda^{n-1} |S|$$

Geodesic balls

Balls in Cartan-Hadamard manifolds satisfy the Euclidean isoperimetric inequality:



$$S \subset M$$

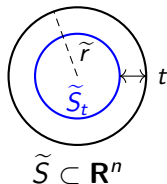
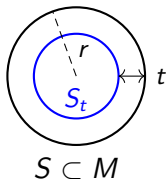


$$\tilde{S} \subset \mathbf{R}^n$$

$$|S| = |\tilde{S}| \implies |B| \leq |\tilde{B}|.$$

Geodesic balls

Proof. Let $S_t, \tilde{S}_t$ be the spheres at distance t inside $S, \tilde{S}$.



Then $r \leq \tilde{r}$, and

$$\begin{aligned} |S_t| &= \left| \frac{r-t}{r} S \right| \leq \left(\frac{r-t}{r} \right)^{n-1} |S| \\ &\leq \left(\frac{\tilde{r}-t}{\tilde{r}} \right)^{n-1} |\tilde{S}| = \left| \frac{\tilde{r}-t}{\tilde{r}} \tilde{S} \right| = |\tilde{S}_t|. \end{aligned}$$

So, by the coarea formula,

$$|B| = \int_0^r |S_t| dt \leq \int_0^r |\tilde{S}_t| dt \leq |\tilde{B}|.$$

Beyond geodesic balls

Why would the above proof not work for convex bodies?

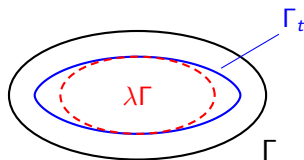
Beyond geodesic balls

Why would the above proof not work for convex bodies?

The proof used that the inner parallel spheres are rescalings:

$$S_t = \frac{r-t}{r} S.$$

For a smooth convex body, the inner parallel hypersurfaces Γ_t are rescalings of Γ only when Γ is a sphere.



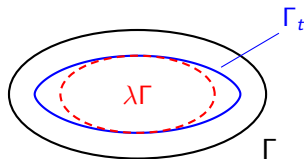
Beyond geodesic balls

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For arbitrary domains, replace the distance from Γ by any function f . The coarea formula $\int_M |\nabla f| = \int_{-\infty}^{\infty} |f^{-1}(s)| ds$ shows that the isoperimetric inequality in M is equivalent to the sharp Sobolev inequality

$$\left(\int_M |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq \frac{1}{n|B^n|^{\frac{1}{n}}} \int_M |\nabla f|.$$

Convexity in Cartan-Hadamard manifolds

Similarities with $\mathbb{R}^n$:

- ▶ Every pair of points is joined by a unique geodesic.
- ▶ The distance function from a convex set is convex.
- ▶ Convex sets have a dimension: their relative interior is a totally geodesic submanifold (Cheeger-Ebin).

Differences from $\mathbb{R}^n$:

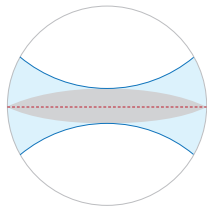
- ▶ The convex hull of 3 points may have interior!
- ▶ The boundary of the convex hull of a set may not contain any geodesic segment (Lytchak-Petrinin): no Carathéodory theorem, no simplices.
- ▶ The distance function from the boundary of a convex body may not be convex in its interior!

Notions of convexity

Equivalent notions of convexity in $\mathbf{R}^n$ diverge in Cartan-Hadamard manifolds:

$$h\text{-convex} \subsetneq d\text{-convex} \subsetneq \text{convex}$$

- ▶ *convex*: geodesics between points of the set lie in the set;
- ▶ *d-convex*: the distance function from the boundary is convex inside the set;
- ▶ *h-convex*: through each boundary point there passes a supporting horosphere.



In $\mathbf{H}^2$: the blue region is convex but not *d-convex*; the gray tube about a geodesic is *d-convex* but not *h-convex*.

Curvature integrals

- ▶ Steiner's formula holds in Cartan-Hadamard manifolds, with “=” replaced by “ $\geq$ ”.
- ▶ Monotonicity fails for the total Gauss-Kronecker curvature: there are convex hypersurfaces γ nested inside Γ with

$$\mathcal{M}_{n-1}(\gamma) > \mathcal{M}_{n-1}(\Gamma) \quad (\text{Dekster, 1981}).$$

- ▶ In $\mathbf{H}^3$, spheres do not minimize the total mean curvature $\mathcal{M}_1$ among convex surfaces of given area (Naveira-Solanes).

So the isoperimetric problem in Cartan-Hadamard manifolds has been open even for convex domains.

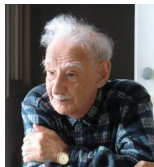
Cartan-Hadamard Conjecture



In 1926 André Weil, who was a student of Hadamard at the time, proved that the isoperimetric inequality holds in Cartan-Hadamard manifolds of dimension $n = 2$.

Method: By the Riemann mapping theorem, a simply connected domain is conformal to a disk, with metric $e^{2u}|dz|^2$, where $\Delta u = -Ke^{2u} \geq 0$. So u is subharmonic, and the inequality follows from an estimate for subharmonic functions.

Cartan-Hadamard Conjecture



In the 1970s and 80s Aubin, Gromov, Burago, and Zalgaller conjectured that the isoperimetric inequality should hold in Cartan-Hadamard manifolds of any dimension $n \geq 2$.

Cartan-Hadamard Conjecture



The case $n = 4$ was proved by Chris Croke in 1984.

Method: Santaló's formula expresses $|\Omega|$ as an integral over the geodesic chords of Ω issuing from Γ . Since geodesics diverge faster than in $\mathbf{R}^n$, Hölder's inequality then yields an isoperimetric inequality in every dimension, whose constant is sharp only for $n = 4$.

Cartan-Hadamard Conjecture



The case $n = 3$ was proved by Bruce Kleiner in 1992.

Method: The isoperimetric profile reduces the problem to a mean curvature estimate for isoperimetric regions. Via convex hulls and the AM-GM inequality, this follows from the total Gauss-Kronecker curvature estimate $\int GK \geq 4\pi$ for convex surfaces, a consequence of the Gauss equation and the Gauss-Bonnet theorem.

Cartan-Hadamard Conjecture

Joint work with Shibing Chen (USTC) and Peng Wang (Fujian Normal U.):

Theorem

Let $\Omega \subset M^n$, $3 \leq n \leq 9$, be a bounded set in a Cartan-Hadamard manifold, with boundary Γ . Then

$$\frac{|\Gamma|^n}{|\Omega|^{n-1}} \geq \frac{|\mathbf{S}^{n-1}|^n}{|B^n|^{n-1}},$$

with equality only if Ω is isometric to a Euclidean ball.

This also gives a unified proof of the results of Kleiner ($n = 3$) and Croke ($n = 4$).

The Isoperimetric Profile Method

Isoperimetric Profile

The *isoperimetric profile* of any Riemannian manifold M is the function $\mathcal{I}_M: [0, |M|) \rightarrow \mathbf{R}$ given by

$$\mathcal{I}_M(v) := \inf \{ |\Gamma| \mid \Omega \subset M, |\Omega| = v, \text{diam}(\Omega) < \infty \},$$

Proving the isoperimetric inequality is equivalent to showing that

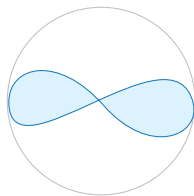
$$\mathcal{I}_M \geq \mathcal{I}_{\mathbf{R}^n},$$

for any Cartan-Hadamard manifold M of dimension n . It suffices to show that

$$\mathcal{I}_B \geq \mathcal{I}_{\mathbf{R}^n}$$

for a family of (open) geodesic balls $B \subset M$ whose radii grow arbitrarily large and eventually cover any bounded set $\Omega \subset M$.

Existence and regularity of isoperimetric regions



Fix a ball $B \subset M$ and consider its *isoperimetric regions*, i.e., sets $\Omega \subset B$ which minimize perimeter for given volume v :

$$|\Gamma| = \mathcal{I}_B(|\Omega|).$$

- ▶ Existence of Ω follows from Blaschke selection principle.
- ▶ Γ is $\mathcal{C}^{1,1}$ near ∂B , and $\mathcal{C}^\infty$ away from ∂B , except for a closed singular set of dimension $\leq n - 8$.
- ▶ Γ has constant mean curvature $H_0 = H_0(v)$, in the interior of B and its mean curvature is $\leq H_0$ on ∂B .

Estimating the mean curvature

Here the mean curvature H is the sum of the principal curvatures. It turns out that

$$\mathcal{I}'_B(v) = H_0(v).$$

Furthermore we know that

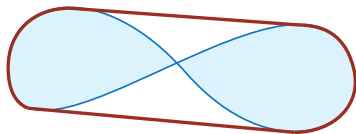
$$\mathcal{I}'_{\mathbf{R}^n}(v) = (n-1) \left(\frac{|\mathbf{S}^{n-1}|}{\mathcal{I}_{\mathbf{R}^n}(v)} \right)^{\frac{1}{n-1}}.$$

So it suffices to show that

$$H_0^{n-1}|\Gamma| \geqslant (n-1)^{n-1}|\mathbf{S}^{n-1}|,$$

i.e., the mean curvature of Γ is at least that of a Euclidean sphere with the same area.

Estimating the Gauss-Kronecker curvature



Let $\Gamma_0 := \partial(\text{conv } \Gamma)$. Since $H \leq H_0$, the AM-GM inequality yields

$$H_0^{n-1} |\Gamma| \geq \int_{\Gamma \cap \Gamma_0} H^{n-1} \geq (n-1)^{n-1} \int_{\Gamma \cap \Gamma_0} GK = (n-1)^{n-1} \int_{\Gamma_0} GK.$$

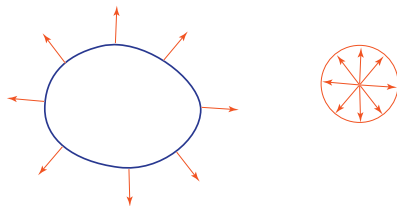
So it would suffice to show that

$$\int_{\Gamma_0} GK \geq |\mathbf{S}^{n-1}|$$

for convex hypersurfaces.

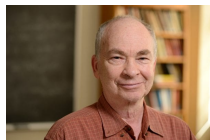
The total curvature inequality

In $\mathbf{R}^n$ the inequality $\int_{\Gamma} GK \geq |\mathbf{S}^{n-1}|$ is trivial: the Gauss map $\nu: \Gamma \rightarrow \mathbf{S}^{n-1}$ of a convex hypersurface is onto, and its Jacobian is GK . So $\int_{\Gamma} GK$ is the volume of the Gauss map, which is $|\mathbf{S}^{n-1}|$.



For $n = 3$ it follows from the Gauss equation and the Gauss-Bonnet theorem (Kleiner). It is open in general for $n \geq 4$.

Total curvature and the isoperimetric inequality



With Joel Spruck, by extending Kleiner's isoperimetric profile method to all dimensions:

Theorem

If $\int_{\Gamma} GK \geq |\mathbf{S}^{n-1}|$ for convex hypersurfaces in M^n , then the isoperimetric inequality holds in M^n .

Recently the total curvature inequality has been established when $-b^2 \leq K \leq -a^2 < 0$:

- ▶ in all dimensions, for convex hypersurfaces with $\text{diam}(\Gamma) \leq \delta_n a/b^2$;
- ▶ in dimensions 4 and 5, for all convex hypersurfaces, if $a^2 \geq \lambda_n b^2$ ($\lambda_4 \approx 0.58$, $\lambda_5 \approx 0.83$).

In particular the isoperimetric inequality holds in pinched Cartan-Hadamard 5-manifolds.

Total curvature inequality

$$\int_{\Gamma} GK \geq |\mathbf{S}^{n-1}|$$

AM-GM

Mean curvature inequality

$$H_0^{n-1}|\Gamma| \geq (n-1)^{n-1}|\mathbf{S}^{n-1}|$$

isoperimetric profile

Isoperimetric inequality

$$\frac{|\Gamma|^n}{|\Omega|^{n-1}} \geq \frac{|\mathbf{S}^{n-1}|^n}{|B^n|^{n-1}}$$

The CMC inequality

In the recent breakthrough, we control the mean curvature directly, bypassing the total curvature inequality:

Theorem

Let $\Gamma \subset M^n$, $3 \leq n \leq 9$, be a smooth compact embedded hypersurface with constant mean curvature $H > 0$. Then

$$|\Gamma| \geq \left(\frac{n-1}{H} \right)^{n-1} |\mathbf{S}^{n-1}|,$$

with equality only if Γ bounds a Euclidean ball.

In $\mathbf{R}^n$ this follows from Alexandrov's theorem: Γ is a sphere of radius $(n-1)/H$.

The inequality persists for isoperimetric regions in B , where $H = H_0$ only on $\Gamma \setminus \partial B$, and $H \leq H_0$ on $\Gamma \cap \partial B$.

The Euclidean Case

The classical route

By scaling we may assume that $H = n - 1$; we need to show that $|\Gamma| \geq |\mathbf{S}^{n-1}|$.

In $\mathbf{R}^n$, Minkowski's formula and the divergence theorem give

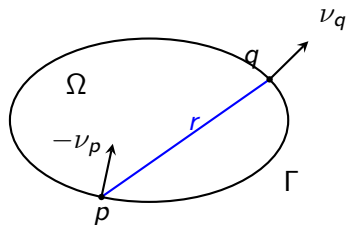
$$\int_{\Gamma} H \langle x, \nu \rangle = (n - 1) |\Gamma|, \quad \int_{\Gamma} \langle x, \nu \rangle = n |\Omega|.$$

So $|\Gamma| = n |\Omega|$, and the isoperimetric inequality yields $|\Gamma| \geq |\mathbf{S}^{n-1}|$.

But this is circular, and there is no position vector x in M .

Instead we use integrals over pairs of points of Γ , in the spirit of Banchoff-Pohl (1971).

Chords



For distinct points $p, q \in \Gamma$ let

$$r := |p - q|, \quad \theta := \frac{q - p}{r}, \quad u := \langle \theta, -\nu_p \rangle, \quad v := \langle \theta, \nu_q \rangle.$$

So u, v are the cosines of the angles between the chord and the inward normal at p , and the outward normal at q .

On the unit sphere $\mathbf{S}^{n-1}$ we have $u = v = r/2$.

Banchoff-Pohl (1971)

For a closed embedded hypersurface $\Gamma \subset \mathbf{R}^n$ bounding Ω ,

$$\int_{\Gamma \times \Gamma} r^{2-n} \geq n |\mathbf{S}^{n-1}| |\Omega|,$$

with equality only for spheres. For $n = 2$ this is the isoperimetric inequality $|\Gamma|^2 \geq 4\pi|\Omega|$.

Their proof expresses the deficit as an integral over $\Gamma \times \Gamma$ of a nonnegative function of the chord. Our approach follows the same template.

To bring in the mean curvature, we integrate the Laplacian on Γ of such weights.

The Minkowski-Green identity

Let $\psi: (0, \infty) \rightarrow \mathbf{R}$ have a pole at 0:

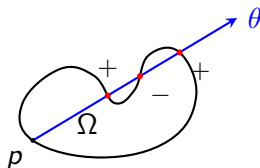
$$\psi'(r) = C_\psi r^{2-n} + O(r^{3-n}).$$

For $q \in \Gamma$ let $\psi_q := \psi(|\cdot - q|)$ on Γ , and let $\tilde{\Delta}_\Gamma \psi_q$ be its Laplacian on $\Gamma \setminus \{q\}$, which depends on H . Near q , ψ_q behaves like a Green function: $\Delta_\Gamma \psi_q = \tilde{\Delta}_\Gamma \psi_q + C_\psi |\mathbf{S}^{n-2}| \delta_q$. So

$$\int_\Gamma \tilde{\Delta}_\Gamma \psi_q = -C_\psi |\mathbf{S}^{n-2}|.$$

For $\psi = r^2/2$ the same computation gives Minkowski's formula.

The degree identity



Fix $p \in \Gamma$, and let $P_p(q) := (q - p)/|q - p|$. A ray from p into Ω alternately exits ($v > 0$) and enters ($v < 0$) Ω :

$$\sum_{q \in P_p^{-1}(\theta)} \operatorname{sgn} v(p, q) = 1_{S_p^-}(\theta),$$

where S_p^- is the hemisphere centered at $-\nu_p$. Since $\operatorname{Jac}(P_p) = v/r^{n-1}$, the area formula applied to u^{n-2} , and symmetrization, yield

$$\int_{\Gamma \times \Gamma} \frac{uv(u^{n-3} + v^{n-3})}{r^{n-1}} = 2^{2-n} |\mathbf{S}^{n-1}| |\Gamma|.$$

Combining the two identities

For constants $\kappa > 0$ and c , the two identities give

$$\int_{\Gamma \times \Gamma} \left(\kappa + \tilde{\Delta}_{\Gamma} \psi_q + \tilde{\Delta}_{\Gamma} \psi_p + c \frac{uv(u^{n-3} + v^{n-3})}{r^{n-1}} \right) = \kappa |\Gamma| (|\Gamma| - L),$$

where L depends only on n , C_{ψ} , κ , and c . So if the integrand is nonnegative, then $|\Gamma| \geq L$.

Ansatz

Set

$$F := r^{n-1}(\kappa + \tilde{\Delta}_\Gamma \psi_q + \tilde{\Delta}_\Gamma \psi_p) + c uv(u^{n-3} + v^{n-3}).$$

For given ψ , κ , c , F is a function of r , u , v alone, i.e., it depends on Γ only through the chord and the normals at its end points.

Find a weight ψ with a pole at 0, and constants $\kappa > 0$ and c , such that

$$F \geq 0$$

for all chords of every closed hypersurface Γ with $H = n - 1$, with equality on the chords of the unit sphere $\mathbf{S}^{n-1}$.

Then $L = |\mathbf{S}^{n-1}|$, and so $|\Gamma| \geq |\mathbf{S}^{n-1}|$.

On the chords of $\mathbf{S}^{n-1}$ we have $u = v = r/2$, and equality there forces ψ_q to be the Green function of the sphere.

Dimension three

In $\mathbf{R}^3$ take

$$\psi(r) := 2 \log r, \quad \kappa = 2, \quad c = 0.$$

Then

$$F = 2((r - u - v)^2 + (u - v)^2) \geq 0,$$

with equality only if $u = v = r/2$. So

$$|\Gamma| - |\mathbf{S}^2| = \frac{1}{2|\Gamma|} \int_{\Gamma \times \Gamma} \frac{F}{r^2} \geq 0.$$

Here the degree identity is not needed.

Dimension five

In $\mathbf{R}^5$ the Green functions of the sphere are generated by

$$\psi(r) := 2 \log r - \frac{2}{r^2}.$$

With $\kappa = 5$, $c = 8$, $\sigma := u + v$, and $\tau := u - v$,

$$5 + \tilde{\Delta}_{\Gamma} \psi_q + \tilde{\Delta}_{\Gamma} \psi_p + 8 \frac{uv(u^2 + v^2)}{r^4} = \frac{F}{r^4},$$

where

$$F = (r - \sigma)^2(5r^2 + 2r\sigma + \sigma^2 + 8) + \tau^2(2r^2 + 8 - \tau^2).$$

Since $|\tau| \leq 2$, $F \geq 0$, with equality only if $u = v = r/2$. So

$$|\Gamma| - |\mathbf{S}^4| = \frac{1}{5|\Gamma|} \int_{\Gamma \times \Gamma} \frac{F}{r^4} \geq 0.$$

Nonpositive Curvature

Jacobi fields

Jacobi fields Z along a geodesic γ are variation fields of families of geodesics:

$$Z'' + KZ = 0,$$

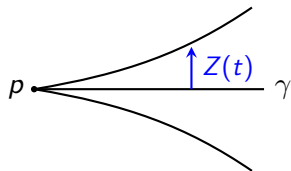
where $K \leq 0$ is the curvature matrix along γ .

If $Z(0) = 0$, then $Z(t) = A(t)Z'(0)$, where

$$A'' + KA = 0, \quad A(0) = 0, \quad A'(0) = I.$$

Since $K \leq 0$, geodesics spread faster than in $\mathbf{R}^n$:

$$J(t) := \det A(t) \geq t^{n-1}.$$



Jacobi fields along chords

In M , the chord from p to q is a geodesic $\gamma: [0, r] \rightarrow M$, and $u := \langle \gamma'(0), -\nu_p \rangle$, $v := \langle \gamma'(r), \nu_q \rangle$. Let $J := \det A(r) \geq r^{n-1}$ along γ .

The curvature enters the identities in two ways:

- ▶ In the degree identity, r^{n-1} is replaced by J .
- ▶ In the Minkowski-Green identity, $\tilde{\Delta}_\Gamma \psi_q$ gains the term $\psi'(r) \operatorname{tr}_\Gamma \mathcal{E}_p$, where

$$\mathcal{E}_p := \nabla^2 r_q(p) - \frac{1}{r} (g - dr_q \otimes dr_q) \geq 0, \quad r_q := d(\cdot, q),$$

is the excess of the Hessian of the distance.

Both identities keep the same masses.

The curved integrand

In dimension 5, replacing r^4 by J in the degree term,

$$5 + \tilde{\Delta}_\Gamma \psi_q + \tilde{\Delta}_\Gamma \psi_p + 8 \frac{uv(u^2 + v^2)}{J} = \frac{\tilde{F}}{r^4},$$

where

$$\tilde{F} = F + 2r(r^2 + 2)(\mathrm{tr}_\Gamma \mathcal{E}_p + \mathrm{tr}_\Gamma \mathcal{E}_q) - 8uv(u^2 + v^2)\left(1 - \frac{r^4}{J}\right).$$

Again $\tilde{F}$ depends only on the chord and the normals at its end points (and on the curvature of M along the chord).

The Hessian excesses enter with a favorable sign, and the Jacobian excess $1 - r^4/J$ with an unfavorable sign. We need to weigh one against the other.

Endpoint comparison

Proposition

For all distinct points $p, q \in \Gamma$,

$$\mathrm{tr}_{\Gamma} \mathcal{E}_p + \mathrm{tr}_{\Gamma} \mathcal{E}_q \geqslant \frac{2|uv|}{r} \left(1 - \frac{r^{n-1}}{J} \right).$$

Positivity in dimension five

Lemma

$\tilde{F} \geq 0$, with equality only if $u = v = r/2$ and $J = r^4$.

If $uv \leq 0$, all terms of $\tilde{F}$ are nonnegative. If $uv > 0$, endpoint comparison shows that the Hessian excess outweighs the Jacobian excess, and an elementary estimate gives $\tilde{F} \geq 0$.

Hence

$$|\Gamma| - |\mathbf{S}^4| = \frac{1}{5|\Gamma|} \int_{\Gamma \times \Gamma} \frac{\tilde{F}}{r^4} \geq 0.$$

Rigidity

If $|\Gamma| = |\mathbf{S}^{n-1}|$, then $\tilde{F} \equiv 0$, so on every chord

$$u = v = \frac{r}{2}, \quad J = r^{n-1}.$$

- ▶ **Convexity:** every geodesic crosses Γ transversally, entering Ω and then exiting it. So it meets Γ at most twice, and Ω is geodesically convex.
- ▶ **Flatness:** $J = r^{n-1}$ forces $|A(t)\xi| = t|\xi|$, so $K = 0$ along every chord, and Ω is flat.
- ▶ **Roundness:** by Alexandrov's theorem, Γ is a round sphere.

Isoperimetric regions

Let $\Omega \subset B$ be an isoperimetric region, scaled so that $H = H_0 = n - 1$ on $\Gamma \setminus \partial B$ and $H \leq H_0$ on $\Gamma \cap \partial B$.

Then $\tilde{F} \geq 0$ still holds, but the integrated identity gains the term

$$\int_{\Gamma \times \Gamma} \psi'(r) \left((n - 1 - H(p))u + (n - 1 - H(q))v \right),$$

which is nonnegative: at $p \in \Gamma \cap \partial B$ the chord points into the convex ball B , so $u \geq 0$.

So $H_0^{n-1}|\Gamma| \geq (n - 1)^{n-1}|\mathbf{S}^{n-1}|$, and the isoperimetric profile argument completes the proof.

Higher Dimensions

Beyond dimension five

With a radial weight $\psi(r)$ and the degree weight u^{n-2} , the scheme works only in dimensions 3 and 5.

In general we let the weight depend also on the angle,

$$\partial_r \psi(r, v) = r^{2-n} (a(r) - b(r)v^2),$$

and use a general degree identity: for every bounded φ ,

$$\int_{\Gamma \times \Gamma} \frac{v\varphi(u) + u\varphi(v)}{J} = 2l_\varphi |\mathbf{S}^{n-2}| |\Gamma|,$$

where $l_\varphi := \int_0^1 \varphi(t) (1 - t^2)^{\frac{n-3}{2}} dt$.

Calibrations

With $H = n - 1$, set

$$\tilde{F} := r^{n-1}(\kappa + \tilde{\Delta}_\Gamma \psi_q + \tilde{\Delta}_\Gamma \psi_p) + \frac{r^{n-1}}{J}(v\varphi(u) + u\varphi(v)).$$

We call (ψ, φ, κ) a *calibration* in dimension n if

$$2|\mathbf{S}^{n-2}|(C_\psi - I_\varphi) = \kappa|\mathbf{S}^{n-1}|,$$

and, in every Cartan-Hadamard n -manifold, $\tilde{F} \geq 0$, with equality only if $u = v = r/2$ and $J = r^{n-1}$.

Theorem

If there is a calibration in dimension $n \geq 3$, then the CMC inequality and the isoperimetric inequality hold in dimension n .

Calibrations in dimensions 3 through 9

n	weights	curvature estimate
3	$\psi = 2 \log r, \varphi = 0$	endpoint comparison
5	$\psi = 2 \log r - 2/r^2, \varphi = 8t^3$	endpoint comparison
4, 6, 7	$a(r) - b(r)v^2, \varphi \geq 0$	endpoint comparison
8, 9	$a(r) - b(r)v^2, \varphi \geq 0$	joint endpoint comparison

- ▶ For $n = 4, 6, 7, 8, 9$, the weights a, b are piecewise polynomial, and positivity reduces to polynomial inequalities, certified by exact Bernstein coefficients (75 certificates, in Mathematica).
- ▶ For $n = 8, 9$, isoperimetric regions may be singular; the identities persist across the singular set, which has dimension $\leq n - 8$.

Outlook

- ▶ Calibrations exist at least up to dimension 13 (computations not included in the paper).
- ▶ In every dimension, the conjecture is now reduced to finding a calibration.
- ▶ The current calibration scheme runs into an obstruction in dimensions above 30.

Thanks for your attention!