

# Total Curvature and Isoperimetric Inequality in Nonpositively Curved Spaces



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The classical isoperimetric inequality states that any bounded set  $\Omega \subset \mathbf{R}^{n \geq 2}$ ,

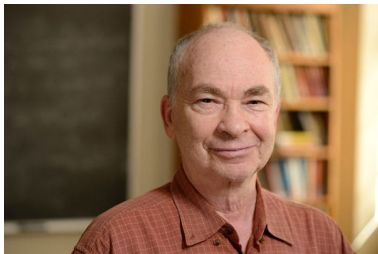
$$\text{per}(\Omega) \geq n \omega_n^{\frac{1}{n}} \text{vol}(\Omega)^{\frac{n-1}{n}},$$

where  $\omega_n$  is the volume of the unit ball.



In this talk we discuss how this inequality may be extended to *Cartan-Hadamard manifolds*, spaces of nonpositive curvature which generalize  $\mathbf{R}^n$ .

This is joint work with Joel Spruck (Johns Hopkins U.).



# Early History and First proofs of the isoperimetric inequality

## Queen Dido of Carthage, 9<sup>th</sup> century B.C.



According to Virgil (*Aeneid*, 29–19 BC) Dido was allowed to claim as much land for Carthage as could be enclosed within an Ox's hide (cut into thin ribbons).

# Queen Dido of Carthage, 9<sup>th</sup> century B.C.



Map of Tunis (modern Carthage) from 1535.

## Archimedes: c. 287 – c. 212 B.C.

- ▶ Computes  $\pi$  (*Measurement of a Circle*, Prop. 3):

$$3\frac{10}{71} \leq \pi \leq 3\frac{1}{7}.$$



- ▶ Computes the area and volume of sphere in his book *On the Sphere and Cylinder*.

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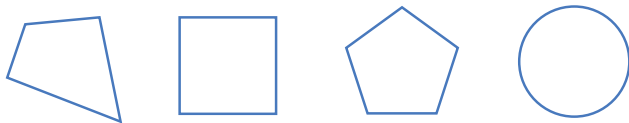


- ▶ Computes the area and volume of sphere in his book *On the Sphere and Cylinder*.



## Zendorus, 150 B.C.

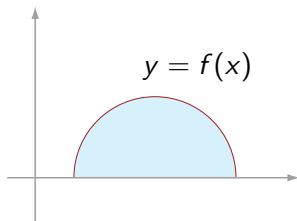
In *On isoperimetric figures* proves the isoperimetric inequality for polygonal curves in the plane.



Reported to have also studied the 3-dimensional problem, but the book is lost (likely it burned with the library of Alexandra in 48 BC).

## Euler 1744

Prompted by the Bernoulli brothers, proves that the area under the graph of a smooth function is maximized by a semicircle.

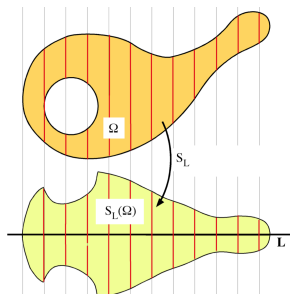


In the process discovers the Euler-Lagrange equation which lays the foundations for *Calculus of Variations*. Let  $\Phi := y + \lambda\sqrt{1 + y^2}$ . Then

$$\Phi_y - \frac{d}{dx}\Phi_{y'} = 0.$$

## Steiner, 1838

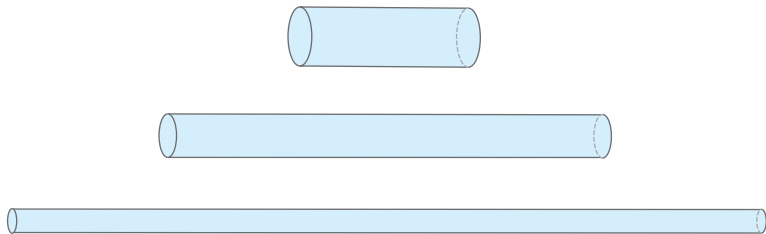
Discover's a symmetrization which preserves volume, and reduces perimeter of any compact domain  $\Omega \subset \mathbf{R}^n$  with regular boundary.



So a perimeter minimizer (if it exists!) must be a sphere.

## Existence of a minimizer

Existence of a minimizer is not obvious, as first pointed out and remedied by Schwartz and Weierstrauss.



A sequence of domains with fixed volume and decreasing perimeter may not have any reasonable limit.

## Existence of a minimizer

A minimizer exists because:

- ▶ Steiner symmetrization does not increase the outradius.
- ▶ **Blaschke Selection Principle:** The space of compact subsets of a metric space is locally compact with respect to Hausdorff distance.



# Applications

## Applications: Sobolev Inequality (with sharp constant)

For  $\Omega \subset \mathbf{R}^n$  and  $f \in \mathcal{W}_0^{1,1}(\Omega)$ ,

$$\left( \int_{\Omega} f^{\frac{n}{n-1}} d\mu \right)^{\frac{n-1}{n}} \leq \frac{1}{n\omega_n^{\frac{1}{n}}} \int_{\Omega} |\nabla f| d\mu,$$

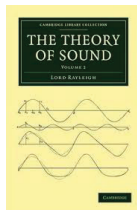
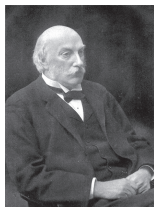
with equality if and only if  $\Omega$  is a ball.

## Applications: Faber-Krahn Inequality

Let  $\lambda_1$  denote the first Dirichlet eigenvalue. Then,

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

where  $B$  is a ball with  $\text{vol}(B) = \text{vol}(\Omega)$ ; furthermore, equality holds only if  $\Omega$  is a ball.



The inequality was proved in 1920s, as had been conjecture by Lord Rayleigh in his book, *Theory of Sound*, in 1877.



Sobolev and Faber-Krahn inequalities both hold on Cartan-Hadamard manifolds provided that the isoperimetric inequality holds there as well. Many other inequalities would follow too.

# The Variational Approach (Curvature and Quermass Integrals)

Symmetrization is a powerful tool in analysis, and can even be used to establish the isoperimetric inequality in the hyperbolic space  $\mathbf{H}^n$ , but it does not work in a general Riemannian space, due to absence of any symmetry or linear structure.

We need a more local or *variational approach* ...

# Brunn-Minkowski (-Lyusternic) Inequality, 1887–1935

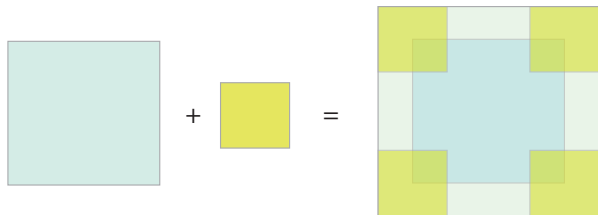


For  $\Omega_1, \Omega_2 \subset \mathbf{R}^n$

$$\text{vol}(\Omega_1 + \Omega_2)^{\frac{1}{n}} \geq \text{vol}(\Omega_1)^{\frac{1}{n}} + \text{vol}(\Omega_2)^{\frac{1}{n}}.$$

## Brunn-Minkowski (-Lyusternic) Inequality, 1887–1935

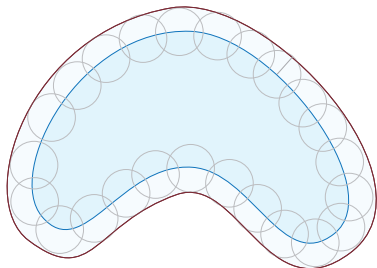
$$\text{vol}(\Omega_1 + \Omega_2)^{\frac{1}{n}} \geq \text{vol}(\Omega_1)^{\frac{1}{n}} + \text{vol}(\Omega_2)^{\frac{1}{n}}.$$



Proof is immediate for rectangles. Follows for all regions by approximation with rectangles and induction (takes only one page!).

# Proof of Isoperimetric Inequality from Brunn-Minkowski

$$\Omega_r := \Omega + rB = \{x \in \mathbf{R}^n \mid \text{dist}(\Omega, x) \leq r\}$$



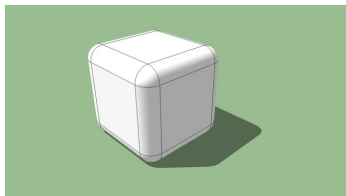
# Proof of Isoperimetric Inequality from Brunn-Minkowski

$$\begin{aligned}\text{vol}(\Omega_r) &= \text{vol}(\Omega + rB) \\ &\geq \left( \text{vol}(\Omega)^{\frac{1}{n}} + r \text{vol}(B)^{\frac{1}{n}} \right)^n \\ &\geq \text{vol}(\Omega) + nr \text{vol}(\Omega)^{\frac{n-1}{n}} \text{vol}(B)^{\frac{1}{n}}\end{aligned}$$

$$\begin{aligned}\text{per}(\Omega) &= \lim_{r \rightarrow 0} \frac{\text{vol}(\Omega_r) - \text{vol}(\Omega)}{r} \\ &\geq n \text{vol}(\Omega)^{\frac{n-1}{n}} \text{vol}(B)^{\frac{1}{n}}\end{aligned}$$

## Steiner Polynomial

If  $\Omega$  is a convex polyhedron in  $\mathbf{R}^3$ , with edge lengths  $L_i$  and corresponding dihedral angles  $\theta_i$ ,



$$\text{vol}(\Omega_r) = \text{vol}(\Omega) + \text{per}(\Omega)r + \left( \sum L_i(\pi - \theta_i) \right) r^2 + \frac{4}{3}\pi r^3.$$

If  $\partial\Omega$  is smooth,

$$\text{vol}(\Omega_r) = \text{vol}(\Omega) + \text{per}(\Omega)r + \left( \frac{1}{2} \int_{\partial\Omega} H \right) r^2 + \left( \frac{1}{3} \int_{\partial\Omega} GK \right) r^3.$$



## Generalized mean curvatures

If  $\Gamma$  is a level set of a function  $f$ , then, at twice differentiable points, the *principal curvatures* of  $\Gamma$

$$\kappa := (\kappa_1, \dots, \kappa_{n-1}),$$

are given by the eigenvalues of the Hessian of  $f$  restricted to the tangent hyperplanes of  $\Gamma$ . We denote the elementary symmetric functions of these curvatures by

$$\sigma_\ell(\kappa) := \sum_{i_1 \leq \dots \leq i_\ell} \kappa_{i_1} \dots \kappa_{i_\ell},$$

and call these the (unnormalized) *generalized mean curvatures* of  $\Gamma$ . In particular,

$$\sigma_1(\kappa) = nH, \quad \text{and} \quad \sigma_{n-1}(\kappa) = GK.$$

## Quermassintegrals

The *quermassintegrals* of  $\Gamma$  are given by

$$V_{n-m}(\Omega) := \frac{(n-m)!(m-1)! \omega_{n-m}}{n! \omega_n} \int_{\Gamma} \sigma_{m-1}(\kappa),$$

for  $m = 1, \dots, n-1$ , and we set  $V_0(\Omega) \equiv 1$ . Then Steiner's polynomial is given by

$$\text{vol}(\Omega_r) = \sum_{m=0}^n V_m(\Omega) r^m.$$

Thus the generalized mean curvatures control the variation of volume.

We will derive a number of estimates, or *Reilly type formulas*, for these quermass integrals on Riemannian manifolds.

# The Cartan-Hadamard Conjecture

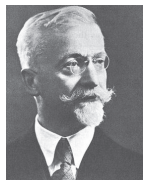
## Cartan-Hadamard manifolds

A Cartan-Hadamard manifold is a complete simply connected manifold of nonpositive curvature.

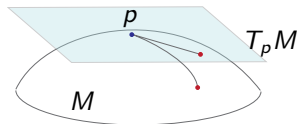


Negatively curved surfaces occur throughout nature. They pack more area inside each geodesic disc, compared to surfaces of zero curvature, through their waves or corrugations.

# Cartan-Hadamard manifolds



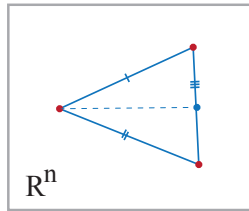
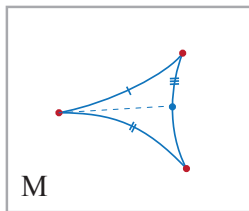
Cartan and Hadamard showed that in a Cartan-Hadamard manifold the exponential map is a diffeomorphism.



So Cartan-Hadamard manifolds are diffeomorphic to  $\mathbf{R}^n$  and every pair of points may be connected with a unique geodesic.

## Cartan-Hadamard manifolds

Nonpositive curvature means that triangles are “thinner” than those in  $\mathbf{R}^n$  (Toponogov’s theorem).



Alternatively, it also means that geodesics emerging from a point diverge faster than those in  $\mathbf{R}^n$ . That rate is measured by Riemann curvature tensor (Jacobi’s Equation).

## Cartan-Hadamard manifolds



$$C = 2\pi D$$



Zero curvature

(a)



$$C < 2\pi D$$



Positive curvature

(b)



$$C > 2\pi D$$



Negative curvature

(c)

Since in a Cartan-Hadamard manifold geodesics diverge faster than in  $\mathbf{R}^n$ , it follows from standard Riemannian comparison theory, that geodesic balls satisfy the isoperimetric inequality.

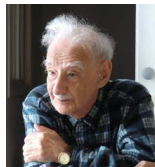
## Cartan-Hadamard Conjecture



In 1927 Andre Weil, who was a student of Hadamard at the time, proved that the isoperimetric inequality holds in Cartan-Hadamard manifolds of dimension  $n = 2$ .



# Cartan-Hadamard Conjecture



In 1970s and 80s Aubin, Gromov, Burago, and Zalgaller conjectured that the isoperimetric inequality should hold in Cartan-Hadamard manifolds of any dimension  $n \geq 2$ .

# Cartan-Hadamard Conjecture



The case  $n = 4$  was proved by Chris Croke in 1984.

# Cartan-Hadamard Conjecture



The case  $n = 3$  was proved by Bruce Kleiner in 1992.

# Cartan-Hadamard Conjecture

The conjecture has remained open in dimensions  $n \geq 5$ .

# Kleiner's Variational Approach

## Isoperimetric Profile

The *isoperimetric profile* of any Riemannian manifold  $M$  is the function  $\mathcal{I}_M: [0, \text{vol}(M)) \rightarrow \mathbf{R}$  given by

$$\mathcal{I}_M(v) := \inf \{ \text{per}(\Omega) \mid \Omega \subset M, \text{vol}(\Omega) = v, \text{diam}(\Omega) < \infty \},$$

Proving the isoperimetric inequality is equivalent to showing that

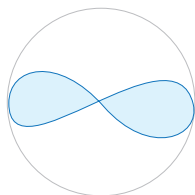
$$\mathcal{I}_M \geq \mathcal{I}_{\mathbf{R}^n},$$

for any Cartan-Hadamard manifold  $M$  of dimension  $n$ . It suffices to show that

$$\mathcal{I}_B \geq \mathcal{I}_{\mathbf{R}^n}$$

for a family of (open) geodesic balls  $B \subset M$  whose radii grows arbitrarily large and eventually covers any bounded set  $\Omega \subset M$ .

## Existence and regularity of isoperimetric regions



Fix  $B \subset M$  and consider its *isoperimetric regions*, i.e., sets  $\Omega \subset B$  which minimize perimeter for given volume  $v$ :

$$\text{per}(\Omega) = \mathcal{I}_B(\text{vol}(\Omega)).$$

- ▶ Existence of  $\Omega$  follows from Blaschke selection principle.
- ▶  $\Gamma := \partial\Omega$  is  $C^\infty$  almost everywhere, and is  $C^{1,1}$  near  $\partial B$ .
- ▶  $\Gamma$  has constant mean curvature  $H_0 = H_0(v)$ , in the interior of  $B$  and its mean curvature is  $\leq H_0$  on  $\partial B$ .

## Estimating the mean curvature

It turns out that

$$\mathcal{I}'_B(v) = (n-1)H_0(v).$$

Furthermore we know that

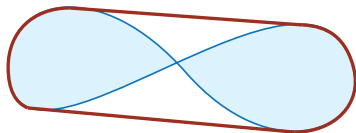
$$\mathcal{I}'_{\mathbb{R}^n}(v) = (n-1) \left( \frac{n\omega_n}{\text{per}(\Omega)} \right)^{\frac{1}{n-1}}.$$

So it suffices to show that

$$H_0^{n-1} \text{per}(\Omega) \geq n\omega_n.$$



## Estimating the Gauss-Kronecker Curvature



Set

$$\Gamma_0 := \partial(\text{conv } \Gamma).$$

By the arithmetic mean versus geometric mean inequality,

$$H_0^{n-1} \text{per}(\Omega) \geq \int_{\Gamma \cap \Gamma_0} H^{n-1} d\sigma \geq \int_{\Gamma \cap \Gamma_0} GK d\sigma = \int_{\Gamma_0} GK d\sigma$$

So, it is left to show:

$$\int_{\Gamma_0} GK d\sigma \geq n\omega_n.$$

# The total curvature inequality

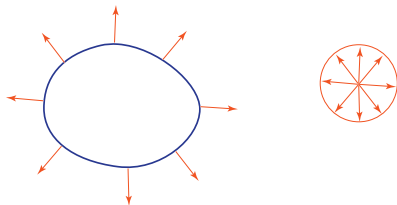
## Problem

For any convex hypersurface  $\Gamma$  in a Cartan-Hadamard manifold  $M^n$ ,  $n \geq 2$ ,

$$\mathcal{G}(\Gamma) := \int_{\Gamma} GK d\sigma \geq n\omega_n.$$

Then the isoperimetric inequality would follow.

- ▶ The total curvature inequality is trivial when  $M = \mathbf{R}^n$ , because  $\mathcal{G}(\Gamma)$  is the volume of the Gauss map.

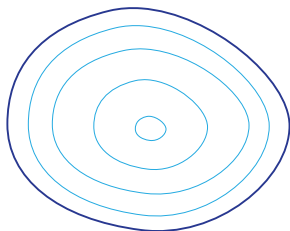


- ▶ Also when  $n = 3$  (the case Kleiner considered) it follows immediately from Gauss-Bonnet theorem, and Gauss's equation.

# The Comparison Formula

## Main idea for proving the total curvature inequality

Shrink  $\Gamma$  without increasing  $\mathcal{G}(\Gamma)$  until  $\Gamma$  collapses to a point.



As all Riemannian manifolds are locally Euclidean to first order, we obtain the desired inequality.

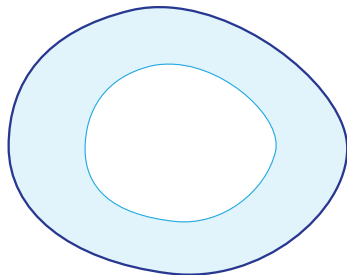
This would be a subtle procedure, because Dekster has constructed nested convex hypersurface  $\gamma$ , and  $\Gamma$  with  $\gamma$  contained inside  $\Gamma$  such that  $\mathcal{G}(\gamma) > \mathcal{G}(\Gamma)$ .

## The comparison formula

We develop a formula for comparing total curvature of level sets of a  $C^{1,1}$  function

$$u: M \rightarrow \mathbf{R}$$

on Riemannian manifolds. Let  $\Gamma$  and  $\gamma$  be regular level sets of  $u$ , with  $\Gamma = \partial\Omega$  and  $\gamma = \partial D$ ,  $D \subset \Omega$ .



How does  $\mathcal{G}(\Gamma)$  compare with  $\mathcal{G}(\gamma)$ ?

## The comparison formula

At every point of  $\Omega \setminus D$  we can find a basis  $E_1, \dots, E_n$  for  $T_p M$  such that

$$E_n = -\frac{\nabla u}{|\nabla u|},$$

and such that the Hessian of  $u$  is given by

$$\nabla^2 u = (u_{ij}) = \begin{pmatrix} |\nabla u| \kappa_1 & & 0 & u_{1n} \\ & \ddots & & \vdots \\ 0 & & |\nabla u| \kappa_{n-1} & u_{(n-1)n} \\ u_{n1} & \cdots & u_{(n-1)n} & u_{nn} \end{pmatrix},$$

where  $\kappa_i$  are principal curvatures of level sets of  $u$ .

## The comparison formula

We define the self-adjoint operator  $\mathcal{T}^u: T_p M \rightarrow T_p M$  by setting

$$(\mathcal{T}_{ij}^u) := \text{cofactor}(u_{ij}).$$

Then

$$GK = \frac{\langle \mathcal{T}^u(\nabla u), \nabla u \rangle}{|\nabla u|^{n+1}},$$

and

$$\text{div} \left( \mathcal{T}^u \left( \frac{\nabla u}{|\nabla u|^n} \right) \right) = \left\langle \text{div}(\mathcal{T}^u), \frac{\nabla u}{|\nabla u|^n} \right\rangle.$$

## The comparison formula

Then Stokes theorem shows that

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = \int_{\Omega \setminus D} \left\langle \operatorname{div}(\mathcal{T}^u), \frac{\nabla u}{|\nabla u|^n} \right\rangle d\mu.$$

Furthermore we can compute that

$$\langle \operatorname{div}(\mathcal{T}^u), \nabla u \rangle = \frac{R(\mathcal{T}^u(\nabla u), E_i, \mathcal{T}^u(E_i), \nabla u)}{\det(\nabla^2 u)},$$

where  $R$  is the Riemann curvature tensor of  $M$  (which vanishes when  $M = \mathbf{R}^n$ ).

Putting everything together, we obtain ...



## The Comparison Formula

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = - \int_{\Omega \setminus D} R_{rnrn} \frac{GK}{\kappa_r} d\mu + \int_{\Omega \setminus D} R_{rkrn} \frac{GK}{\kappa_r \kappa_k} \frac{u_{nk}}{|\nabla u|} d\mu$$

# The Comparison Formula

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = - \int_{\Omega \setminus D} R_{rn rn} \frac{GK}{\kappa_r} d\mu + \int_{\Omega \setminus D} R_{rk rn} \frac{GK}{\kappa_r \kappa_k} \frac{u_{nk}}{|\nabla u|} d\mu$$

## Notes:

- ▶ When  $M$  has constant curvature,  $R_{rn rk} = 0$ . So the second integral vanishes, and we obtain

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = - \int_{\Omega \setminus D} R_{rn rn} \frac{GK}{\kappa_r} d\mu.$$

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- ▶ In addition we may always assume that  $u$  is convex, or  $\kappa_i \geq 0$ .  
So

$$G(\Gamma) \geq G(\gamma)$$

when  $M = H^n$ . So the total curvature inequality holds in  $H^n$ .

## The Comparison Formula

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = - \int_{\Omega \setminus D} R_{rnrn} \frac{GK}{\kappa_r} d\mu + \int_{\Omega \setminus D} R_{rkrn} \frac{GK}{\kappa_r \kappa_k} \frac{u_{nk}}{|\nabla u|} d\mu$$

### Notes:

- ▶ When  $|\nabla u|$  is constant on level sets of  $u$ , or  $u_{nk} = 0$ , then the second integral again vanishes, and we obtain

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = - \int_{\Omega \setminus D} R_{rnrn} \frac{GK}{\kappa_r} d\mu.$$

- ▶ If in addition level sets of  $u$  are convex we obtain

$$G(\Gamma) \geq G(\gamma).$$

So the total curvature inequality holds when  $\Gamma$  is a ball.

# Problems

So where are we going to find a function  $u$  with  $|\nabla u| = \text{const}$  and level sets of  $u$  convex?

*There is only one possibility:* the distance function of  $\Gamma$ .

But

- ▶ Problem #1: In a Cartan-Hadamard manifold the (signed) distance function of a convex hypersurface needs not be convex.
- ▶ Problem #2: The distance function always develops singularities.

## The distance function

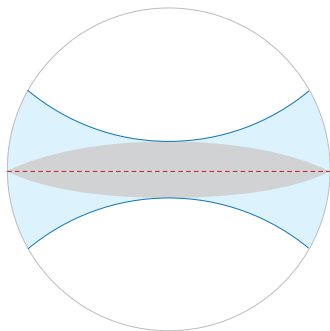
Let  $d: M \times M \rightarrow \mathbf{R}$  denote the distance function of  $M$ . Then the unsigned distance function of  $\Gamma$  is

$$d_{\Gamma}(x) := d(\Gamma, x),$$

and the signed distance function is given by setting  $\widehat{d}_{\Gamma}(x) = -d_{\Gamma}(x)$  if  $x \in \Omega$  and  $\widehat{d}_{\Gamma}(x) = d_{\Gamma}(x)$  otherwise.

## The Convexity Problem

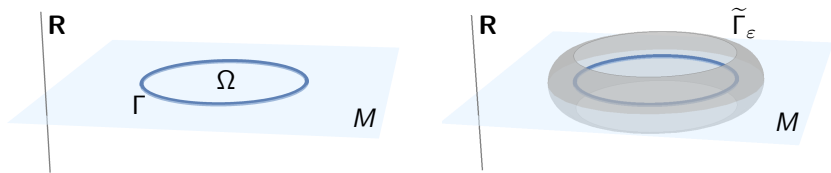
If  $\Gamma$  is a convex hypersurface in a Cartan Hadamard manifold  $M$ , then  $\widehat{d}_\Gamma$  is always convex outside  $\Omega$ , but may not be convex on  $\Omega$ .



In general  $\widehat{d}_\Gamma$  is convex on  $\Omega$  only when  $\Omega$  is *h-convex*, i.e., through each point of  $\Gamma$  there passes a supporting horosphere.

## The convexity problem

We replace  $\Gamma$  by the outer parallel hypersurface  $\tilde{\Gamma}_\varepsilon$  of  $\Omega$  in  $M \times \mathbf{R}$ .



It turns out that

$$\frac{\mathcal{G}(\tilde{\Gamma}_\varepsilon)}{(n+1)\omega_{n+1}} \rightarrow \frac{\mathcal{G}(\Gamma)}{n\omega_n}.$$

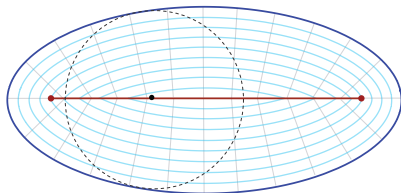
In particular, if  $\mathcal{G}(\tilde{\Gamma}_\varepsilon) \geq (n+1)\omega_{n+1}$ , then  $\mathcal{G}(\Gamma) \geq n\omega_n$ .

So we may assume that  $\hat{d}_\Gamma$  is convex. Then we say that  $\Gamma$  is  $d$ -convex.



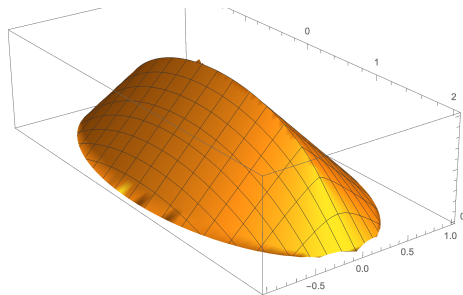
## The singularity problem

Level sets of  $\widehat{d}_\Gamma$  develop singularities as soon as they hit the *cut locus*,  $\text{cut}(\Gamma)$ .



$\text{cut}(\Gamma)$  is the closure of the *medial axis*, i.e., the set of points in  $\Omega$  where the maximal geodesic sphere in  $\Omega$  centered at that point touches  $\Gamma$  multiple times.

## The singularity problem



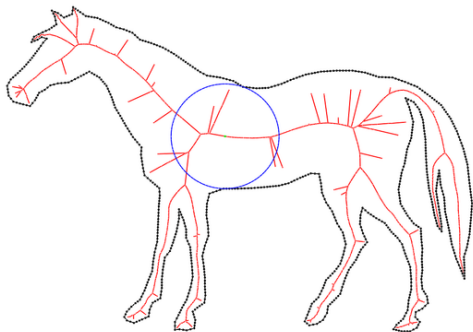
The graph of the distance function resembles a sand dune, and the cut locus corresponds to the ridge.

## The singularity problem



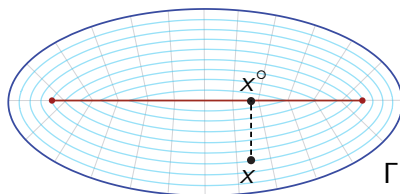
The resemblance is not accidental!

## The singularity problem



The study of medial axis is of interest in pattern recognition.

## The singularity problem



### Theorem (Structure of the cut locus)

For every  $x \in \Omega$ , let  $x^\circ$  be a closest point (or footprint) on  $\text{cut}(\Gamma)$ , then

$$d(x^\circ, \Gamma) \geq d(x, \Gamma).$$

# Approaching the Singularity Problem via Smoothing the Distance Function

## Possible plan for dealing with singularities

- ▶ **Smooth** the distance function  $u := \widehat{d}_\Gamma$  by a one parameter family of functions  $v^r$ .
- ▶ Multiply  $v^r$  by a **cutoff function**  $\eta$  in a neighborhood of the singularities, away from a fixed point  $x_0$  on  $\text{cut}(\Gamma)$ .
- ▶ Apply the comparison formula to level sets of  $\eta v^r$  (**integrate by parts**)
- ▶ Deviation of  $|\nabla v^r|$  from 1 introduces a number of **error terms**.
- ▶ Show that the error terms vanish or become nonnegative as  $r \rightarrow 0$ .

## The inf-convolution

the *inf-convolution* (or more precisely *Moreau envelope* or *Moreau-Yosida regularization*) of a function  $u: M \rightarrow \mathbf{R}$  is given by

$$\tilde{u}^t(x) := \inf_y \left\{ u(y) + \frac{d^2(x, y)}{2t} \right\}.$$

$\tilde{u}^t$  is the unique viscosity solution of the Hamilton-Jacobi equation

$$f_t + \frac{1}{2} |\nabla f|^2 = 0$$

for functions  $f: \mathbf{R} \times M \rightarrow \mathbf{R}$  satisfying the initial condition

$$f(0, x) = u(x).$$

When  $M = \mathbf{R}^n$ , epigraph of  $\tilde{u}^t$  is the Minkowski sum of the epigraphs of  $u$  and  $|\cdot|^2/(2t)$

The point where the infimum is achieved is called the *proximal point*.

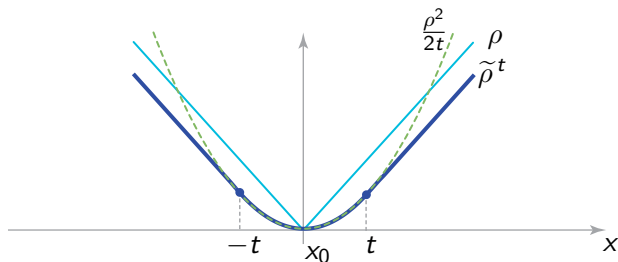


## The inf-convolution

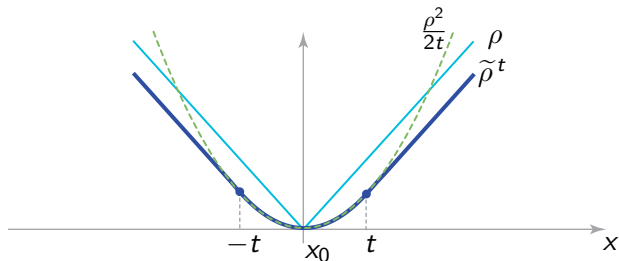
A simple but highly illustrative example of inf-convolution occurs when it is applied to  $\rho(x) := d(x_0, x)$ , the distance from a single point  $x_0 \in M$ . Then

$$\tilde{\rho}^t(x) = \begin{cases} \rho^2(x)/(2t), & \text{if } \rho(x) \leq t, \\ \rho(x) - t/2, & \text{if } \rho(x) > t, \end{cases}$$

which is known as the *Huber function*.



## The inf-convolution

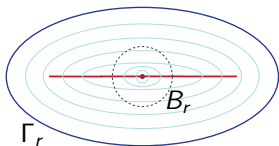
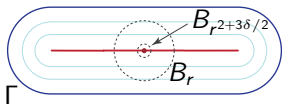


$\tilde{\rho}^t$  is  $\mathcal{C}^{1,1}$  and convex,  $\inf(\tilde{\rho}^t) = \inf(\rho)$ ,  $|\nabla \tilde{\rho}^t| \leq 1$  everywhere,  $|\nabla \tilde{\rho}^t| = 1$  when  $\rho > t$ , and  $|\nabla^2 \tilde{\rho}^t| \leq C/t$ .

We show that all these properties are shared by the inf convolution of  $\hat{d}_\Gamma$  when  $\Gamma$  is  $d$ -convex.

## Smoothing the distance function

We pick  $x_0 \in \text{cut}(\Gamma)$ , set  $0 < \delta < 2/3$ , and approximate  $u := \widehat{d}_\Gamma$  by a family of  $C^{1,1}$  functions  $v^r$  such that



- ▶  $v^r \rightarrow u$ , as  $r \rightarrow 0$
- ▶  $v^r$  has a single minimum point at  $x_0$
- ▶  $v^r$  is *radial* on  $B_{r+3\delta/2}$
- ▶  $v^r$  coincides with the inf-convolution of a perturbation of  $u$  outside  $B_r$ .

The zero level set of  $v^r$  will be called  $\Gamma_r$  and the domain it bounds  $\Omega_r$ . We will have

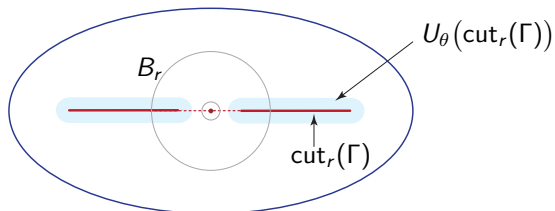
$$\mathcal{G}(\Gamma_r) \rightarrow \mathcal{G}(\Gamma).$$

# The Cut Off Function

Let

$$\text{cut}_r(\Gamma) := \text{cut}(\Gamma) \setminus B_{r/2}$$

and  $\eta$  be a smooth cutoff function with  $\eta \equiv 0$  on  $U_\theta \text{cut}_r(\Gamma)$  and  $\eta \equiv 1$  on  $U_{2\theta} \text{cut}_r(\Gamma)$



We set  $\theta := Cr^{1+\delta/2}$  (so  $\theta/r \rightarrow 0$  as  $r \rightarrow 0$ ).

## The Error Terms

Applying the comparison formula to  $\eta v^r$  and integration by parts yields:

$$\mathcal{G}(\Gamma_r) - \mathcal{G}(\gamma) = \text{I}(\Omega_r \setminus D) + \text{II}(\Omega_r \setminus D) + \text{III}(\Omega_r \setminus D),$$

where

$$\begin{aligned} \text{I}(\cdot) &:= \int_{(\cdot)} \left( \eta_k \frac{GK}{\kappa_k} \frac{v_{nk}^r}{|\nabla v^r|} - \eta_n GK \right) d\mu, \\ \text{II}(\cdot) &:= \int_{(\cdot)} \eta R_{\ell k \ell n} \frac{GK}{\kappa_\ell \kappa_k} \frac{v_{nk}^r}{|\nabla v^r|} d\mu, \\ \text{III}(\cdot) &:= - \int_{(\cdot)} \eta R_{\ell n \ell n} \frac{GK}{\kappa_\ell} d\mu. \end{aligned}$$

We have  $\text{III} \geq 0$ . Furthermore it can be shown that  $\text{II}$  vanishes, via Reilly type formulas. But controlling  $\text{I}$  does not appear to be easier than the original problem.

# Other Applications of the Comparison Formula

## Theorem

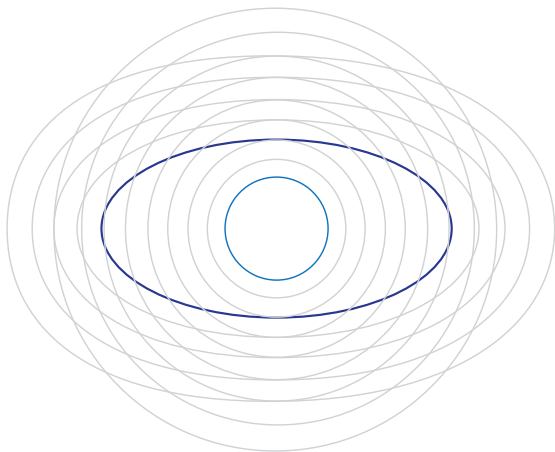
Let  $\Gamma, \gamma$  be convex hypersurfaces in a Cartan-Hadamard manifold  $M$ . Suppose that the sectional curvature of  $M$  is constant near  $\Gamma$ . Then

$$\mathcal{G}(\Gamma) \geq \mathcal{G}(\gamma).$$

We set

$$u = u^\lambda := r^2 + \lambda\rho$$

where  $r$  is distance from  $\Omega$  and  $\rho$  is distance from  $D$ .

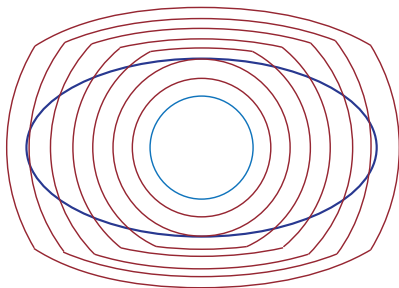




We set

$$u = u^\lambda := r^2 + \lambda\rho$$

where  $r$  is distance from  $\Omega$  and  $\rho$  is distance from  $D$ .



Fix  $\epsilon > 0$ . Let

$$\Omega_\lambda := \{u^\lambda < \epsilon^2\} \quad \text{and} \quad \Gamma_\lambda := \partial\Omega_\lambda.$$

Then

$$\mathcal{G}(\Gamma_\lambda) - \mathcal{G}(\gamma) = I(\Omega_\lambda \setminus D) + II(\Omega_\lambda \setminus D).$$

Recall that  $I \geq 0$  always.

To show that  $\mathbb{I} = 0$  note that

$$\mathbb{I}(\Omega_\lambda \setminus D) = \mathbb{I}(\Omega_\lambda \setminus \Omega) + \mathbb{I}(\Omega \setminus D).$$

We may choose  $\lambda$  so small that  $\Gamma_\lambda \subset U$ . Then

$$\mathbb{I}(\Omega_\lambda \setminus \Omega) \subset \mathbb{I}(U) = \int_U R_{rkrn} \frac{GK}{\kappa_r \kappa_k} \frac{w_{nk}}{|\nabla w|} d\mu = 0,$$

since  $R_{rkrn} = 0$  on  $U$ . Furthermore,

$$\mathbb{I}(\Omega \setminus D) = \int_{\Omega \setminus D} R_{rkrn} \frac{GK}{\kappa_r \kappa_k} \frac{u_{nk}}{|\nabla u|} d\mu = \int_{\Omega \setminus D} R_{rkrn} \frac{GK}{\kappa_r \kappa_k} \frac{\rho_{nk}}{|\nabla \rho|} d\mu = 0,$$

since  $u = \lambda \rho$  and  $\rho_k = 0$  on  $\Omega \setminus D$ . So

$$\mathbb{I}(\Omega_\lambda \setminus D) = 0.$$

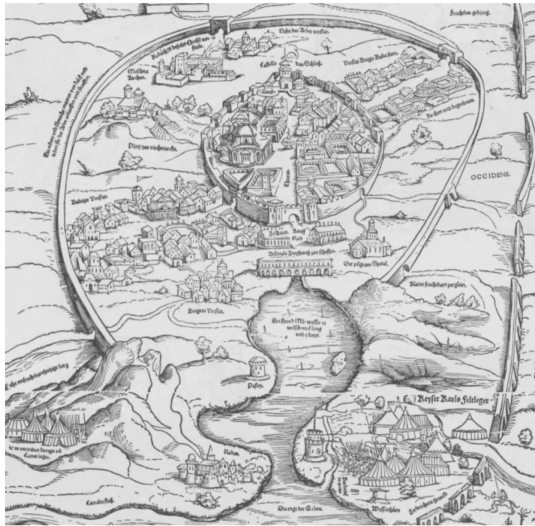
## Corollary

*If the curvature of a Cartan-Hadamard manifold  $M$  vanishes outside a compact set, then  $M = \mathbf{R}^n$ .*

## Proof.

Let  $\Gamma$  be a convex hypersurface enclosing the region with vanishing curvature. Since the curvature vanishes near  $\Gamma$ , Gauss-Bonnet-Chern theorem yields that  $\mathcal{G}(\Gamma) = n\omega_n$ . So all convex hypersurfaces inside the region enclosed by  $\Gamma$  must have total curvature at most  $n\omega_n$ , including all balls, which yields that curvature must vanish inside  $\Gamma$ . □

The above result had been established earlier by Greene and Wu.



Thanks for your attention!