A Cartan–Hadamard space $M$ is a complete simply connected Riem. mfd with non-positive curvature.

Example: $\mathbb{R}^n \& \mathbb{H}^n$

Basic properties:

- $M$ is diffeo. to $\mathbb{R}^n$
- Every pair of points can be joined by a unique geodesic
- CH-mflds are $CAT(0)$ Spaces.

geodesic metric spaces
where triangles are "thinner" than those in $\mathbb{R}^n$. 
Another way to think of nonpositive curvature:

- Exp. map is expansive.

\[ d(\lambda x, \lambda y) \leq \lambda d(x, y) \]
$|S| \leq |S'|$

Balls in $\mathbb{CH}$-fields satisfy the Euclidean isoperimetric inequality.

$S \subset M$

$\tilde{S} \subset \mathbb{R}^n$

$|S| = |	ilde{S}| \implies |B| \leq |\tilde{B}|$

**Proof**

$r \leq \tilde{r}$
\[ |S_t| = \frac{r-t}{r} |S| \]
\[ \leq \left( \frac{r-t}{r} \right)^{n-1} |S| \]
\[ \leq \left( \frac{\tilde{r}-t}{\tilde{r}} \right)^{n-1} |\tilde{S}| \]
\[ = \frac{\tilde{r}-t}{\tilde{r}} |\tilde{S}| \]
\[ = |\tilde{S}_t| \]

\[ |B| = \int_0^r |S_t| \, dt \leq \int_0^r |\tilde{S}_t| \, dt < |\tilde{B}| \]

co-area formula

Exercise: Why would not the above proof work for convex bodies?

\[ \text{CH- Conjecture (Aubin, Brémaud, Burago, Zalgaller)} \]

The Euclidean isoperimetric holds in CH-maths.
\[ |\Omega| = |\mathcal{S}| \Rightarrow |\Omega| \leq |\mathcal{B}| \]

CH- Conjecture

**Known for:**

- \( H^n \) (e.g. Steiner symmetrization)
- \( n=2 \) (Weil), \( n=3 \) (Kleiner), \( n=4 \) (Croke)
- Large volumes, when \( K<0 \) (Vau, Bang-Zabel)
- Small volumes (Johnson-Magen)
- For every \( n \) there is a const \( C_n \) s.t.
  \[ \frac{|\Omega|^{n-1}}{|\Gamma|^n} \leq C_n \]  
  (Spruck-Hoffman)

Equivalent to Sobolev Ineq. (By the eccentric family)
\[
\left( \sum_{\Omega} + \right) \leq \frac{1}{n/\mathcal{B}^n} \mathcal{D}^f \mathcal{M}
\]

Open even in the convex case

- Convexity is not so convenient in the absence of linear structure.

A couple of Proofs of the Classical Isop. Ineq. in \( \mathbb{R}^n \)

Steiner Symmetrization

(requires existence of a minimizer)

Brunn–Minkowski Ineq.
\[ |\Omega_1 + \Omega_2|^n \geq |\Omega_1|^n + |\Omega_2|^n \]

Proof is immediate for rectangles. Follows for all regions by approximation.

\[ \Omega_r := \Omega + rB = \{ x \in \mathbb{R}^n \mid \text{dist}(\Omega, x) \leq r \} \]

unit ball

\[ |\Omega_r| = |\Omega + rB| \geq (|\Omega|^{\frac{1}{n}} + r|B|^{\frac{1}{n}})^n \geq |\Omega| + nr|\Omega|^{\frac{n-1}{n}}|B|^{\frac{1}{n}} \]

\[ |\Gamma| = \lim_{r \to 0} \frac{|\Omega_r| - |\Omega|}{r} \geq n|\Omega|^{\frac{n-1}{n}}|B|^{\frac{1}{n}} \]
Convexity in CH-mfds

- Similarities with \( \mathbb{R}^n \):
  
  * distance function from a convex set is convex
    
    (a function \( f: M \to \mathbb{R} \) is convex if its restriction to geodesics is convex)

  * Convex sets have dimension
    
    (their relative interior is a totally geodesic submfd, Cheeger-Ebin)

- Differences from \( \mathbb{R}^n \):
  
  * convex hull of 3 points may have interior!
boundary of a convex body
might not be convex in
the interior!

* Boundary of the convex hull
  of a set may not contain
  any geodesic segment!
  (no Carathéodory theorem)
  (no simplices)

* Equivalent notions of convexity
  in $\mathbb{R}^n$ diverge in CFT-molds.

$h$-convex $\subseteq$ $d$-convex $\subseteq$ convex

through each
boundary point
there passes
a horosphere

dist function
from boundary
is convex
inside the set.

geod. between
points are
contained
in the set.
More strange things will be mentioned later.

Example

\[ H^2 \]

\[ H^2 \]

c-convex but not d-convex

tube about a geodesic: d-convex, but not h-convex.

So how are we going to study the isop. problem in CH-mfdgs?

- We can adopt a variational approach, in the sense of Steiner.
which leads to integrals of generalized mean curvature (called quermassintegrals or mixed volumes in $\mathbb{R}^n$).

Then we can study to what extent various Alexander-Fenchel type inequalities hold.

Let us begin by reviewing

**Steiner Polynomial**, **Mean curvature integrals**, and **Alexander-Fenchel-Sseg in $\mathbb{R}^n$**.

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**Steiner’s Polynomial**

How mean curvatures enter the picture

- If $\mathcal{P} \subset \mathbb{R}^n$ is a smooth closed embedded hypersurface
\[ l_t := l + t v \]

is well-defined for small \( t \)

\[
|P_t| = |P| + C_1 M_1(P) t + \ldots + C_{n-1} M_{n-1}(P) t^{n-1}
\]

Steiner formula

total (first) mean curvature

\( M_r(p) := \sum_{P} \tau_r(K) \)

\[ K = (K_1, \ldots, K_{n-1}) \]

principal curvatures

\[ \tau_r(K) = \sum_{i_1 \ldots i_r} K_{i_1} \ldots K_{i_r} \]

down symmetric functions
$S_0 := 1$ (by convention)

So $M_0 = |P|$

Proof [Steiner's polynomial]

$f! P \rightarrow \mathbb{R}^t$

$f(x) := x + tv$

$d f = I + t \, dv$

Ki are the eigen values of $dv$

$|P_t| = \int_P \det (d f)$

$= \int_P \det (I + t \, dv)$
Crefton's formula for total mean curv.

If \( \Gamma \) is convex, \( M_r(\Gamma) \) have a more geometric description:

\[
M_r(\Gamma) := \sum_{V \in \text{Gr}(n-1-r,n)} |\Pi_r(\Gamma)|
\]

Average size of projections of \( \Gamma \) into \( \text{cn}(n-1-r) \)-dim subspaces.

Eq: \( M_{n-2} \) is the mean width.

\[
M_{n-2}(\Gamma) := |\Omega|
\]
So if \( \mathcal{S} \) is nested inside \( \mathcal{P} \), then

\[
M_k(\mathcal{S}) \leq M_k(\mathcal{P})
\]

**Monotonicity Formula**

**Alexandrov-Fenchel Inequalities**

\[
\frac{M_k(\mathcal{P})^{n-k}}{M_{k-1}(\mathcal{P})^{n-k-1}} \geq C_{n,k} \frac{M_k(\mathcal{S}^{h-1})^{n-k}}{M_{k-1}(\mathcal{S}^{h-1})^{n-k-1}}
\]

**Examples**

\[
\begin{align*}
\text{A} & \quad k = 0 \quad \text{(Isop. Ineq.)} \\
\frac{|T|^n}{|T^{n-1}|} & \quad \geq \quad \frac{|S^{h-1}|^n}{(B^h)^{n-1}}
\end{align*}
\]
\[ \frac{M_1(C)^{n-1}}{|P|^{n-2}} \geq \frac{M_1(S^{n-1})^{n-1}}{|S^{n-1}|^{n-2}} \]

among convex hypersurfaces with the same area sphere (only sphere) minimizes total (1st) mean curvature.

In particular, for \( n = 3 \)

\[ M_1(C) \geq \sqrt[3]{16\pi |P|} \]

\[ M_{n-1}(C) \geq |S^{n-1}| \]

the volume of the Gauss map
Unto CH-molds

* Steiner polynomial holds in CH-molds

with "=" replaced by ».

* Alexander Fenchel Ineq. have been extended to H^n, via harmonic curvature flow, but not always in their sharpest form. (Anders, Hu, Li, Wang, Xia, ...)

* Sharp Minkowski ineq. is not known even in H^3!
Almost all fundamental questions are open in CH-molds:

- Isop. Ineq. is open (CH conjecture)

- Gauss-Kronecker ineq is open
  (not even known that $M_{n-1}(R^n) > 3!$)

- Minkowski ineq is open
  (for $K < a < 0$)
  (But for $K \leq 0$, solved recently) in dim 3

\[ G^- \text{ ineq} \rightarrow \text{Isop. Ineq.} \]

\[ \text{Minkowski ineq.} \xrightarrow{\text{G-S}} \text{Isop. Ineq.} \]

assuming d-concavity
* Warning: More strange convexity phenomena in C^1-mflds:

- Monotonicity fails for \( M_{n-1} \):
  \[ M_{n-1}(\emptyset) \neq M_{n-1}(\emptyset) \]

**Delešter's Example (JP, 1981)**

\[ \mathbb{H}^2 \]

Warped product of \( \mathbb{H}^2 \)'s

---

**Nareira-Solanes Example**

\[ M_1(\mathbb{R}) \] in \( \mathbb{H}^3 \) is not
minimal biholomorphic
Suppose Minkowski ineq. holds for $d$-convex hypersurfaces $P \subset M$, i.e.

\[ M_1(P) \geq M_1(S) \]

Sphere in $\mathbb{R}^n$ with $|S| = |P|$ and "" equal only if $P$ bounds an $n$-dimensional Euclidean ball.

Then isop. ineq.

holds in $M$ for $d$-convex hypersurfaces.
Ingredients of the proof

Reach (in the sense of Federer)

Reach of a convex hypersurface $\Gamma \subset M$, is the sup radii of balls which roll freely inside $\Gamma$.

\[ \text{reach}(\Gamma) = \text{dist} \left( \Gamma, \text{cut}(\Gamma) \right) \]

Inner and outer parallel hypersurfaces

Let $\Gamma^+_t$, $\Gamma^-_t$ be the inner and outer parallel hypersurfaces (level sets) of $\Gamma$. 
\[ t \leq \text{reach}(\mathbb{P}) \Rightarrow \left| (\mathbb{P}_+)_t \right| = \left| \mathbb{P} \right| \]
\[ t > \text{reach}(\mathbb{P}) \Rightarrow \left| (\mathbb{P}_-)_t \right| < \left| \mathbb{P} \right| \]

Coarea formula

\[ f : \Omega \rightarrow [a, b], \text{ Lipschitz} \]

\[ \left| \nabla f \right| = 1 \]

\[ \left| \Omega \right| = \int_a^b \left| f^{-1}(t) \right| dt \]

Proof:

Let \( S \) be a sphere in \( \mathbb{R}^n \) with \( |S| = \left| \mathbb{P} \right| \). Then
Inradius (P) ≤ radius (S)

We know that Isop Ineq holds for spheres. So suppose that P is not a sphere. Then

\[ M_1(P) > M_1(S) \]

⇒ \[ |P_t| < |S_t| \], for small \( t > 0 \) (\( \blacklozenge \))

If \( |P_t| ≤ |S_{t+}| \), for all \( t \in [0, \text{inradius}(P)] \)

we are done by the coarea formula.

Suppose then that

\[ |P_{t_0}| > |S_{t_0}| \]

for some \( t_0 \).

Let

\[ \bar{s} := \sup \{ s \mid |(P_{s+})_t| > |S_{s+t}| \} \].
Then

\[ |(\Gamma_{-t_0})_S| = |S_{-t_0 + \bar{s}}| \] (**) 

\[ \Rightarrow \quad M_1((\Gamma_{-t_0})_S) > M_1(S_{-t_0 + \bar{s}}) \]

\[ \Rightarrow \quad \bar{s} = t_0 \quad \text{(otherwise we can push higher)} \]

There are now two cases to consider:

1. \( t_0 > \text{reach}(\Gamma) \)
2. \( t_0 \leq \text{reach}(\Gamma) \)

**Case 1**

\[ |(\Gamma_{-t_0})_S| < |\Gamma_{-t_0 + s}| = |\Gamma| = |S| = |S_{-t_0 + \bar{s}}| \]

\( \text{violates (**)} \)

**Case 2**
\[ |(\Gamma_{t+t})_s| = |\Gamma_{t+t+s}|, \forall s \in [0, t] \]

\[ \text{riodnets (\ast)} \]

**Afterword**

We have actually proved something stronger than the isoperimetric inequality:

1. \[ \Omega \]
2. \[ \Gamma \subset M \]
3. \[ S \subset S_t \]

\[ |\Omega| = |S| \]

- \( \Lambda_t \): Annular domain between \( \Gamma \) and \( \Gamma_t \)
- \( A_t \): “ “ “

\[ |\Lambda_t| \leq |A_t| \]

for all \( t \in [0, \text{inrad}(S)] \)

In particular,

\[ |\Omega| \leq |A_t| \]
Bonnesen style

isop. Ineq.

Questions

★ Can one prove this annular isop. ineq. without Mink. Ineq?

★ Even in $H^n$, or $\mathbb{R}^n$?

★ Does it just follow from the main isop. ineq.?

★ In dimensions 3, 8, 4 we already have the isop. ineq. in CT1-mflds. So can we get the Mink. Ineq out of that?
How to compare $M_r(\mathcal{C})$ $\&$ $M_r(\mathcal{B})$ in $\mathcal{C}$ $\setminus$ $m$-hulls.

\[ \alpha: M \rightarrow \mathbb{R}, \quad C'_{11} \]

\[ \forall u \neq 0 \quad \text{on} \quad \mathbb{S} - \mathcal{D} \]

\[ u \equiv \text{const.} \quad \text{on} \quad \mathcal{F} \cup \mathcal{B} \]

\[ K^n = (K^u_1, \ldots, K^u_{n-1}) \quad \text{principal curvatures} \]

\[ E_1, \ldots, E_{n-1} \quad \text{principal directions} \]

\[ M_r(\mathcal{C}|_{\mathcal{M}} - M_r(\mathcal{B}) = (r+1) \int_{\mathbb{S} - \mathcal{D}} \sigma_{r+1}(K^n) \]

\[ + \int_{\mathbb{S} - \mathcal{D}} (-\sum_{i=1}^{n-1} k^u_i \cdot k^u_{n-i} \cdot k^u_{n} + \frac{1}{|\Delta n|} \sum_{i=1}^{n-2} k^u_i \cdot k^u_{i+1} \cdot \left( \text{R}_{i, n-1} \cdot k^u_{n} \right) ) \]
The "good term"

\[ \Delta \omega_i := D E_i \left( D \omega \right) \]

\[ R_{ijkl} = R(E_i, E_j, E_k, E_l) \] (Riemann tensor)

\[ K_{ij} := R_{ijij} \] (Sectional curvature)

Note: For \( r = 0 \),

\[ \left| \mathcal{P} \mathcal{L} \mathcal{B} \mathcal{L} - r \right| = 2 \int_{\Omega \setminus D} \mathcal{O}_1(K^u) \]

This is a well-known formula which follows from Stokes' theorem.

Because

\[ \mathcal{O}_1(K^u) = \text{div} \left( \frac{Da}{|D\omega|} \right) \]

Proof of the Comparison Formula
**Proof 1:** Divergence of Newton Operators

developed by Reilly

\[ P(\nabla u) := \text{div} \left( \mathbf{\nabla} \nabla \ n - \mathbf{D}^2 u \right) \]

Hessian of \( u \)

\[ Tr^u := \text{Truncation of } P(\mathbf{D}^2 u) \text{ by} \]

\text{removing terms of order higher than } r.

\[ \Rightarrow \quad Tr^u (K^u) = \frac{\left< \text{Tr}(\mathbf{D}u), \mathbf{D}u \right>}{1 |\mathbf{D}u|^r} \]

\[ \text{div}(\text{Tr}^{u-1}(\frac{\mathbf{D}u}{|\mathbf{D}u|^r})) \]

\[ = \left< \text{div}(\text{Tr}^{u-1}), \frac{\mathbf{D}u}{|\mathbf{D}u|^r} \right> + r \frac{\left< \text{Tr}^u (\mathbf{D}u), \mathbf{D}u \right>}{1 |\mathbf{D}u|^{r+2}} \]

\text{Divergence identity for Newton operators. Integrating this formula yields the}
Proof 2: Chern's Formulas

$\sigma_r(K) = \Phi_r(E_1, \ldots, E_{n-1})$

Chern-type forms

$\Phi_r = \sum_{\epsilon_1, \ldots, \epsilon_{n-1}} \omega_{n}^{\epsilon_1} \wedge \cdots \wedge \omega_{n}^{\epsilon_{n-1}} \wedge \theta^{i_1} \wedge \cdots \wedge \theta^{i_{n-1}}$

the sum ranges over $1 \leq i_1, \ldots, i_{n-1} \leq n-1$

with $i_1 < \cdots < i_r$ and $i_{r+1} < \cdots < i_{n-1}$

$\theta^i(E_j) = \delta^i_j$

dual one-forms $\theta$ to $E_i$

$\omega^i_j(\cdot) := \langle D(\cdot) E_i, E_j \rangle$

connection one forms

$\omega^i_j(r \cdot, \cdot)$
\[ \Phi_r(\tilde{\alpha}) = \text{Stokes thm} \]

\[ d \Phi_r = (-1)^{k-1} (r+1) \Phi_{r+1} \Lambda \Theta^n + (-1)^{k-1} \sum \varepsilon(i_1 \cdots i_{n-1}) \omega_{i_1} \Lambda \cdots \Lambda \omega_{i_{n-1}} \Lambda \omega_{i_k} \Lambda \theta_i \Lambda \cdots \Lambda \theta_{i_{n-1}} \]

\[ \Omega^i_j (\cdot, \star) := -< R(\cdot, \star) E_i, E_j > \]

\[ \Omega^i_n (E_n) = \frac{< D E_n (D u), E_i >}{|D u|} = \frac{\text{uni}}{|D u|} \]

**Applications of the Comparison Formula**

The "bad term" (involving mixed curvatures) drops out when...
- \( R = 1 \)
- \( K_m = \text{const} \)
- \( \gamma, \zeta \) are parallel.

Leading to several new inequalities.

- Let \( M \) be a CH-mfd

\[ \& \quad \xi, \theta \subset C^1 \text{ convex hypersurfaces,} \]

then we have the following applications.

\[ \text{A. Monotonicity for total mean curvature:} \]

\[ M_1(\gamma) - M_1(\theta) = 2 \int_{\Omega} \omega(k^n) - (n-1) \int_{\Omega} \text{Ric}(\frac{1}{\omega}) \]

\[ \geq 0 \]

\[ \Rightarrow K \leq \alpha \]

\[ M_1(\gamma) \geq -(n-1)\alpha |\Omega| \]
(generalizes Gallego-Soares in $H^3$)

* Monotonicity for parallel Surfaces

If $P, S$ are parallel (and $S$ is still smooth, i.e., within reach($P$))
then $10u_i = 0$. So

$$M_r((P - M_r(S)) \geq (r+1) \int_{\Omega_D} \sigma_{r+1}(k^n) - a(\nu - 1) \int_{\Omega_D} \sigma_{r-1}(k^n) \geq 0$$

(generalizes Schroeder-Strake)

for $r = n-1$

* Monotonicity for constant curvature:

$$M_r(P) - M_r(S) = (r+1) \int_{\Omega_D} \sigma_{r+1}(k^n) - a(\nu - 1) \int_{\Omega_D} \sigma_{r-1}(k^n) \geq 0$$

(C established earlier by Soares,)

Sharp for $n=3$.
* Rigidity than for curvature

**Thm:** Let $\Gamma$ be a st. convex in

$CH$-mld $M^3$ with $K \leq a \leq 0$

Suppose $K = a$ on $\Gamma$. Then

$K \equiv a$ on $\Omega$

(Refines Gromov, Green-Wu, Ziller, Shroeder,
Strake, Seshadri, for $a = 0$)

- Generalized very recently by G. Petraxin
  to hypersurfaces with semi-def. 2nd fund. form
  in all dimensions.

**Proof:** After replacing $\Gamma$ by another
convex surface in $\Omega$ may assume that
$K \leq a$ at some point in every ngbd of $\Gamma$ in $\Omega$. 
Let $\Omega$ be an inner parallel surface.

Choose $\delta$ so close to $P$ s.t. $K < a$ at some point of $\delta$.

By the comparison formula

$$M_2(\Omega) - M_2(\delta) \geq -a \int_{\Omega} \Sigma_1 = -a(|P| - |\delta|)$$

$$\Sigma_2 = K_\Omega - K_M$$

Gauss' Eq.

$$\Rightarrow M_2(\Omega) = \int_{\Omega} \Sigma_2$$

Gauss-Bonnet Thm

$$= \int_{\Omega} K_\Omega - \int_{\Omega} K_M$$

$$= 4\pi - \int_{\Omega} K_M$$

$$= 4\pi - a |P|$$

So

$$4\pi - a |P|$$
\[ M_2(x) \leq 4n - a|\beta| \]

But
\[ M_2(x) = 4n - 3\kappa \geq 4n - a|\beta| \]

So
\[ M_2(x) = 4n - a|\beta| \]

\[ \Rightarrow k = a \text{ on } 8 \]

Contradiction.

Minkowski inequality in CH-3 mfld

\[ M_1([\gamma]) \leq \sqrt{16n^2 - 2a|\gamma|^2} \]

- Sharpest ineq. known in CH-3 mflds, even H
- Sharp for \( a = 0 \)
Examples in $H$ satisfy

$$M_1(r) > \sqrt{\frac{16n l^2 r^2 + 6 l^4 r^4}{4}} 
> 2.47$$

So if the sharp Minkowski inequality is of the form

$$M_1(r) > \sqrt{16nl^2 (1 - \lambda) l^2}$$

then

$$2 < \lambda < 2.47$$

**Proof:** uses harmonic mean curvature flow

the only flow known to deform convex hypersurfaces to a point in CH-molds.

$$X : \mathbb{R} \times [0, T) \to M \ , \ X_t(\cdot) := X(\cdot, t)$$

$$X_t'(p) = -F_t(p)\nu_t(p) \ , \ X_0(p) = p$$
\[ F_+ := \left( \frac{1}{k_1^+} + \cdots + \frac{1}{k_{n-1}^+} \right)^{-1} \]
\[ = \frac{S_{n-1}^+}{\sigma_{n-2}} \]

\[ \Phi(\tau) := \frac{M_2(\mathcal{P}_+)}{16 \Omega(\mathcal{P}_+) + 2a \Omega(\mathcal{P}_+)^2} \]

We compute \( \Phi' \) as follows:

By the comparison & coarea formula:

\[ M_1'(\mathcal{P}_+) = -2 \int_{\mathcal{P}_+} \left( \frac{\sigma_{n-1}^+}{\sigma_{n-2}^+} - \text{Ric}(\mathcal{P}_+) \right) F_+ \, d\mu \]
\[ \leq -2 \int_{\mathcal{P}_+} \left( \frac{\sigma_{n-1}^+}{\sigma_{n-2}^+} \right) \, d\mu + 2a \int_{\mathcal{P}_+} \frac{\sigma_{n-1}^+}{\sigma_{n-2}^+} \, d\mu \]
\[ 1 \mathcal{P}_+ = -M_2(\mathcal{P}_+) \]

\[ \Phi' + 1 = 2 M_1(\mathcal{P}_+) M_1'(\mathcal{P}_+) - 16 \Omega(\mathcal{P}_+) \]

\[ + 4a \Omega(\mathcal{P}_+) \]

\[ \leq 4 M_2(\mathcal{P}_+) \left( -M_2(\mathcal{P}_+) + 4 \Omega \right) \]
$$J_2^+(p) = K_{g_+}(p) - K_M(T_p \mathbb{R}^n)$$

Gauss' Equation

$$\int_{\mathbb{R}^n} K_p = 4\pi$$

Gauss-Bonnet

$$\Rightarrow \quad \mathcal{N}_2(\mathbb{R}^n) > 4\pi - \alpha |\mathbb{R}^n|$$

$$\Rightarrow \quad \mathcal{O}' \leq 0$$

But \( \lim \mathcal{O}(\mathbb{R}^n) = 0 \)

\( t \to 0 \)

\[ \mathcal{O}(\mathbb{R}^n) > 0 \]

\[ \square \]

Note: If \( p \) is h-cone, we can show that...
\[ M_1 (p) \geq \sqrt{16 \|p\|^2 - \frac{7}{2} \|p\|^4} \]

But it is not known if HMCF preserves h-convexity.