Convexity & Rigidity of hypersurfaces in Cartan-Hadamard manifolds

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$M^n(k)$, CH mfld, $K_M \leq k \leq 0$

$\Gamma \subset M^n(k)$, immersed, II_p semidefinite

Theorem: Suppose $K_M \equiv k$ on $T_p\Gamma$

$\Rightarrow$ Γ bounds convex region with $K_M \equiv k$

$$GK := \det(\mathrm{II}_p)$$

$$\mathcal{G}(\Gamma) := \int_\Gamma GK \,, \quad \tilde{\mathcal{G}}(\Gamma) = \int_\Gamma |GK|$$

Corollary: $\Gamma \subset H^3(k)$, $\pi_1(\Gamma) = 0$

conj of Gromov ($k=0$) 1985

$$\boxed{\tilde{\mathcal{G}}(\Gamma) \geq 4\pi - k|\Gamma|} \quad (\star)$$

with equality iff Γ bounds a convex region with $K_H \equiv k$.

History

1898 - Hadamard: $\Gamma \subset \mathbb{R}^3$, $GK > 0 \Rightarrow$ convexity

1957 - Chern-Lashof: $\Gamma \subset \mathbb{R}^3$, $GK \geq 0 \Rightarrow$ convexity

also showed:

$\tilde{\mathcal{J}}(\Gamma) \geq 4\pi$, "=" only for convex.

More generally: $\tilde{\mathcal{J}}(\Gamma) \geq 2\pi(2+2g)$

If "=" holds, Γ is called tight

1966 - Wilmore-Saleemi conjecture:

$\tilde{\mathcal{J}}(\Gamma) \geq 2\pi(2+2g)$ in H

1985 - Gromor

$\tilde{\mathcal{J}}(\Gamma) \geq 4\pi$ in H.

1989 - Shreeder-Strake

Prove Gromor for Γ st. convex

2007 - Solanes

Wilmore-Saleemi is False for $g > 1$.

For any $g > 1$, there is $\Gamma \to \mathbb{H}^3$

with $\tilde{g}(\Gamma) \leq 4\pi + \varepsilon$!

Proof of Corollary

$$\boxed{GK(p) = K_\Gamma(p) - K_H(T_p\Gamma)}$$

Gauss' Eq

$$\tilde{g}(\Gamma) \geq g(\Gamma) = 4\pi - \int_\Gamma K_H(T_p\Gamma) \geq 4\pi - k|\Gamma|$$

If "=" holds in (★), then "=" hold above, so

$$\tilde{g}(\Gamma) = g(\Gamma) \Rightarrow GK \geq 0 \Rightarrow II_\Gamma \text{ is semi-def.}$$

$$\int_\Gamma K_H(T_p\Gamma) = k|\Gamma| \Rightarrow K_H \equiv k \text{ on } T_p(\Gamma)$$

So Γ bounds a convex body with $K_H \equiv k$ by the main thm. □

Proof of the thm: (Based on an outline by Petrunin)

① Use Gauss & Codazzi-Mainardi Eq

to isometrically embedd Γ in $S^n(k)$

with the same 2nd fund. form.

Model space of const. curvature k

$$M^n(k) \supset \Gamma \xrightarrow{f} \Gamma' \subset S^n(k)$$

② It follows that Γ is convex by theorems of do Carmo–Warner & S. Alexander

③ Show that f may be extended to convex bodies bounded by Γ' & Γ via

- Schur's Arm Lemma
- Reshetnyak's majorization thm
- Kirszbraun's extension thm (generalized to Riem mflds by Lang & Schroeder)

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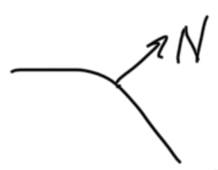
① Isometric immersion into $S^n(k)$ preserving II_p

$$K(X,Y) := \frac{\langle R(X,Y)Y, X\rangle}{|X \times Y|^2}, \quad A(X) := -\nabla_X N$$

$$II_p := \langle A(X), Y\rangle$$



Lemma: Let X, Y be orthonormal vectors tangent to Γ,

N be normal: $\Rightarrow R(X,Y)N = 0.$

$$\langle R(X,Y)Y, X\rangle = k$$

b/c $k \leq 0$

$$\langle R(X+tN, Y)Y, X+tN\rangle \leq k|X+tN|^2 \leq k$$

$$0 = \frac{1}{2}\frac{d}{dt}\Big|_{t=0} \langle R(X+tN, Y)Y, X+tN\rangle$$

$$= \langle R(X,Y)Y, N\rangle$$

$$0 = \langle R(X, Y+Z)(Y+Z), N\rangle$$

$$= \cancel{\langle R(X,Y)Y, N\rangle} + \langle R(X,Y)Z, N\rangle$$

$$+ \langle R(X,Z)Y, N\rangle + \cancel{\langle R(X,Z)Z, N\rangle}$$

$$\Rightarrow \langle R(X,Y)Z, N\rangle = \langle R(Z,X)Y, N\rangle$$

$$\langle R(X,Y)Z, N\rangle = -\langle R(Y,X)Z, N\rangle = \langle R(Z,Y)X, N\rangle$$

$$\Rightarrow \langle R(X,Y)Z, N\rangle = \langle R(Y,Z)X, N\rangle = \langle R(Z,X)Y, N\rangle$$

$\Rightarrow$ By the first Bianchi Identity $\langle R(X,Y)Z, N\rangle = 0.$

$$\overset{\shortparallel}{-\langle R(X,Y)N, Z\rangle}$$

$$(X \wedge Y) := \langle Y, Z\rangle X - \langle X, Z\rangle Y$$

$$\begin{cases} R_\Gamma(X,Y)Z = (R(X,Y)Z)^T + (A(X)\wedge A(Y))Z \\ (R(X,Y)N)^T = D_X A(Y) - A(D_Y X) \end{cases}$$

Gauss & Codazzi-Mainardi Eq.

$$R(X,Y)N = 0 \Rightarrow \begin{cases} (R(X,Y)Z)^T = R(X,Y)Z \\ (R(X,Y)N)^T = 0 \end{cases}$$

$$\Rightarrow \begin{cases} R_\Gamma(X,Y) = k\, X\wedge Y + A(X)\wedge A(Y) \\ D_X A(Y) = A(D_Y X) \end{cases}$$

Exactly the Gauss & Codazzi Eq. for a hypersurface in $S^n(k)$

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② Convexity of Γ & Γ'

By do Carmo - Warner Γ' is convex

$\Rightarrow$ $\mathrm{II}_{\Gamma'}$ is positive semidefinite

$\Rightarrow$ II_{Γ} is positive semidefinite

$\Rightarrow$ Γ is convex by a Alexander

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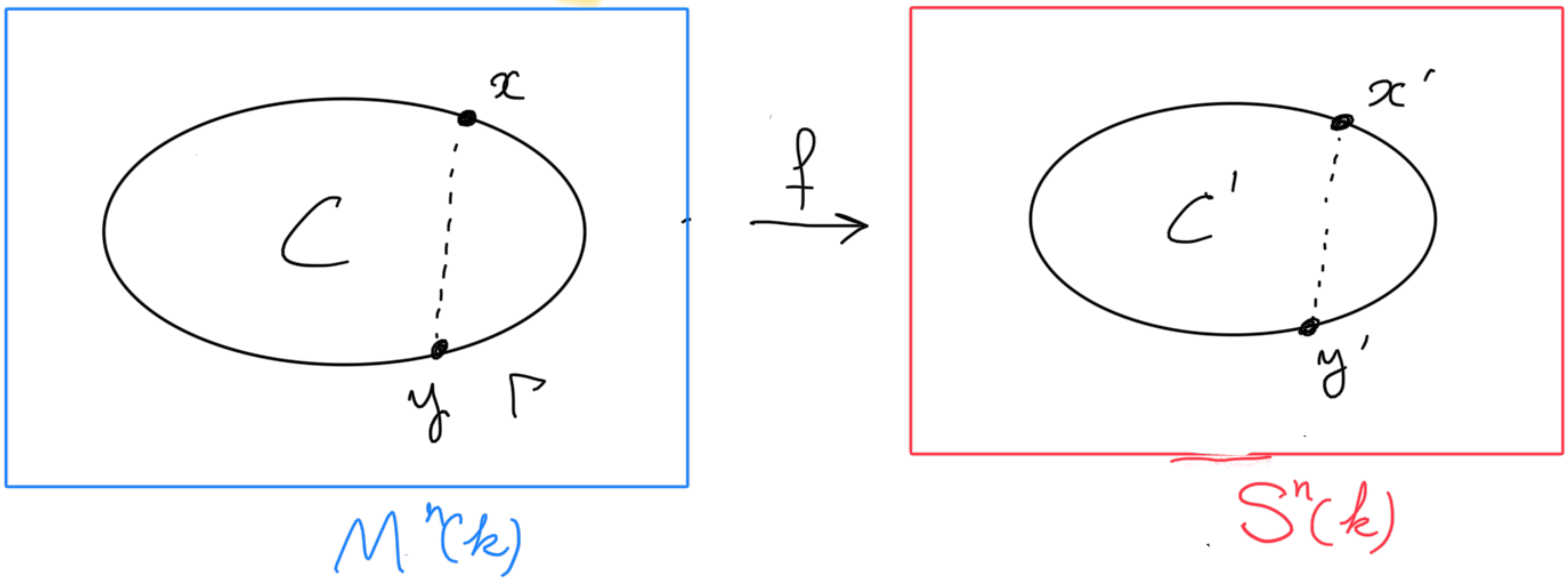
③ Extension of f to convex bodies

bounded by Γ & Γ'.



$\widehat{x'y'}$ arc in the intersection of Γ' with a totally geodesic plane passing through x', y'.

$\widehat{xy} := f^{-1}(\widehat{x'y'})$

geodesic curv. of $\widehat{xy}$ = geod curv of $\widehat{x'y'}$

b/c 1st & 2nd fund forms are the same

$\Rightarrow$ by Schur's Arm Lemma: (generalized to $M^n(k)$)

$$d_M(x,y) \geq d_S(x',y')$$

(f is nonexpansive)
or 1-Lipschitz

- By Reshetnyk's thm f^{-1} is also

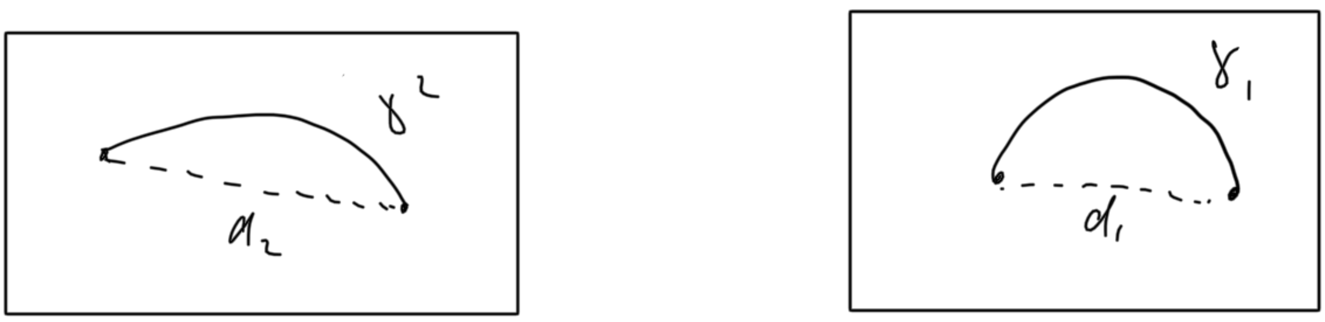
nonexpansive

- So $f: \Gamma \to \Gamma'$ preserves distances in the ambient space:

$$d_S(x',y') = d_M(x,y)$$

- By <u>Kirszbraun</u> f^{-1} extends to a non-expansive map between C' & C
- f^{-1} is an isometry between C' & C.

Generalized Schur Comparison Thm



$M^n(k)$ $S^2(k)$

$$K_2 \le K_1 \Rightarrow d_2 \geq d_1$$

Reshetnyak's Majorization thm:

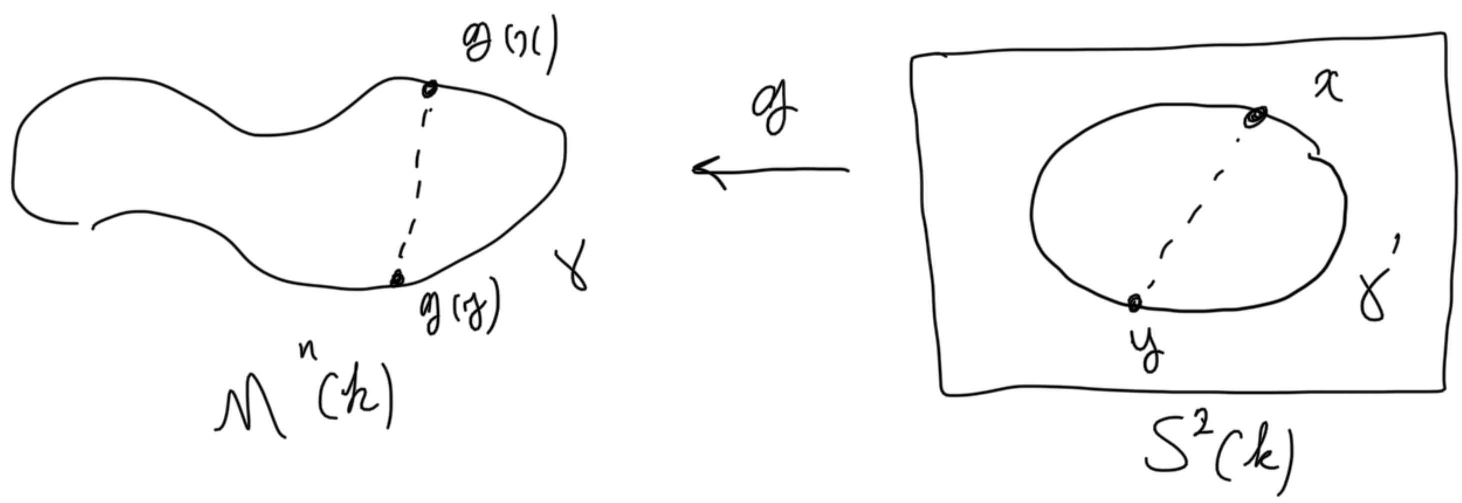
$\gamma \subset M^n(k)$ is a closed

$\exists \; \gamma' \subset S^2(k)$ & $g: \gamma' \to \gamma$

convex

s.t. g preserves arc-length

& g is nonexpansive in the ambient spaces



$$d_M(g(x), g(y)) \leq d_S(x,y)$$

Kirszbrann's Extension thm

Any 1-Lipshitz map = nonexpansive

$$f : X \subset S^n_{(k)} \longrightarrow M^n(k)$$

extends to a 1-Lipshitz map

$$\bar{f} : S^n \longrightarrow M^n$$

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since $\bar{f} : \Gamma' \longrightarrow \Gamma$ preserves path lengths & does not reduce chord-lengths

$\Rightarrow$ $\bar{f} : C' \to C$ is an isometry

Δ-ineq.

So we proved Gromov's conjecture that

$$g(\Gamma) \geq 4R$$

(with "=" $\Leftrightarrow$ Γ bounds convex body)

for $g = 0$ (simply connected surfaces).

The proof can be generalized to all genus g if

- $GK \geq 0 \Rightarrow$ convexity (generalization of Chern-Lashof to CH manifolds)