# Relative Isoperimetric Inequality Outside Convex Bodies

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Joint work with

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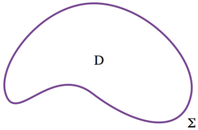
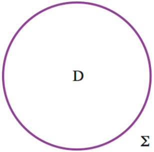
Let $D \subset \mathbf{R}^n$ be a bounded set with boundary $\Sigma := \partial D$ of finite area.




**The Classical Isoperimetric Ineq:**

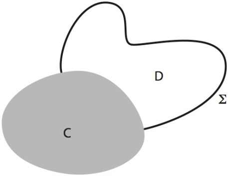
$$\text{Area}(\Sigma) \geqslant \text{Area}\left(\text{sphere of volume}\,\text{Vol}(D)\right).$$

Equality holds only when $D$ is a ball.




("Area" means the $(n-1)$-dimensional, and "Volume" is the $n$-dimensional, Hausdorff measures).

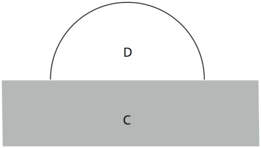
Let $C \subset \mathbf{R}^n$ be a convex set with interior points and smooth ($\mathcal{C}^\infty$) boundary, and $D \subset \mathbf{R}^n - C$ be a bounded set with finite perimeter, and set $\Sigma := \overline{\partial D - C}$.




**Theorem [Relative Isoperimetric Ineq.]**

$$\text{Area}(\Sigma) \geqslant \text{Area}\left(\text{hemisphere of volume}\,\text{Vol}(D)\right).$$

Equality holds only when $D$ is a half-ball.



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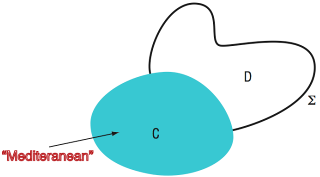


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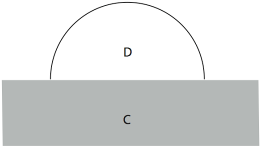

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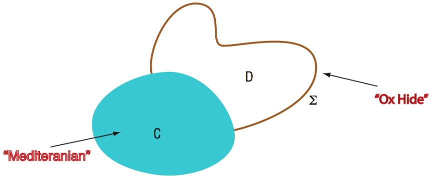


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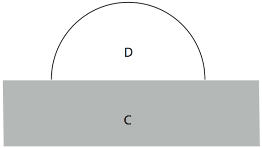
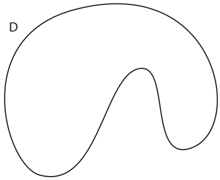


Note: Relative Isop. Ineq $\Longrightarrow$ Classical Isop. Ineq.




- Given a region $D \subset \mathbf{R}^n$, there exists a hyperplane $H \subset \mathbf{R}^n$ which cuts $D$ into regions of equal volume.
- By the Rel. Isop. Ineq., the area of $\partial D$ on either side of $H$ is no less than that of a hemisphere with half the volume of $D$.
- So the area of $\partial D$ cannot be less than that of a sphere with the same volume.

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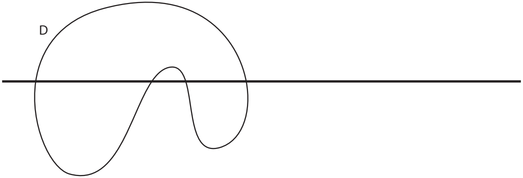


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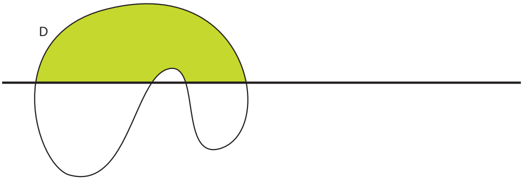


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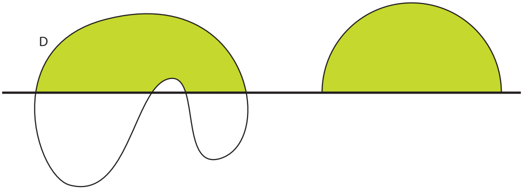


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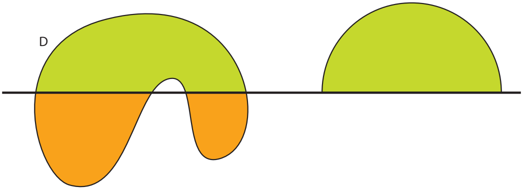


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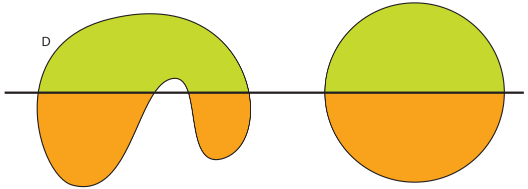


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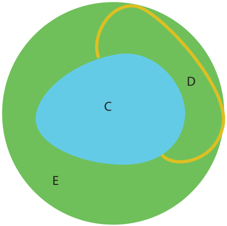
The proof of the relative isoperimetric inequality involves 3 parts:

1. Existence and Regularity of the minimizer
2. Reducing the problem to a curvature estimate
3. Obtaining that estimate.

## Lemma (Existence and Regularity)

*Let $E$ be the closure of a bounded domain with smooth boundary in $\mathbf{R}^n - C$. Then, for any $v \in (0, \mathrm{Vol}\, E)$, there is a set $D \subset E$ of volume $v$ minimizing the area of $\Sigma$.*




Moreover ....

## Lemma (Continued)

(i) *$\Sigma$ has constant mean curvature and is smooth in the interior of $E$ except for a singular set of Hausdorff dimension less than or equal to $(n-8)$.*

(ii) *$\Sigma$ meets $\partial C$ orthogonally except for a singular set of Hausdorff dimension less than or equal to $(n-8)$. In fact $\Sigma$ is smooth at every point of $\Sigma \cap C$ away from this singular set.*

(iii) *If $\partial E - C$ is strictly convex, then $\Sigma$ meets $\partial E - C$ tangentially and it is $\mathcal{C}^{1,1}$ in a neighborhood of $\partial E - C$.*

(iv) *At every point $p \in \Sigma$ there is a tangent cone obtained by blowing up $\Sigma$ about $p$. If this tangent cone is contained in a half space of $\mathbf{R}^n$, then it is the half space and $\Sigma$ is regular at $p$.*

## Note

The $\mathcal{C}^{1,1}$ regularity of $\Sigma$ almost everywhere is enough for our purposes since, by Rademacher's Theorem, a $\mathcal{C}^{1,1}$ hypersurface has principal curvatures defined almost everywhere, and thus we will be able to talk about integral estimates for curvature of $\Sigma$.

To prove the relative isoperimetric inequality we constrct an *exhaustion* of $\mathbf{R}^n$: Fix $p_0 \in \partial C$ and let

$$E_m := \overline{B(p_0, m) - C}.$$

The *isoperimetric profile* of $E_m$ is defined as

$$I_{E_m}(v) := \inf \left\{ \text{Area}(\Sigma) : D \subset E_m, \ \text{Vol}\, D = v \right\}$$

All we need is to show that, for all $v \in (0, \text{Vol}(E_m))$

$$\boxed{I_{E_m}(v) \geqslant I_{\mathbf{H}^n}(v)}$$

where $I_{\mathbf{H}^n}(v)$ denotes the isoperimetric profile of the halfspace $\mathbf{H}^n$:

$$I_{\mathbf{H}^n}(v) := \text{Area}(\text{Hemisphere of volume } v)$$

$$\boxed{I_{E_m}(v) \geqslant I_{\mathbf{H}^n}(v)}$$

It is well known that $I_{E_m}$ is increasing so $I_{E_m}^{-1}$ exists. The above inequality is then equivalent to

$$\boxed{I_{E^m}^{-1}(a) \leqslant I_{\mathbf{H}^n}^{-1}(a)}$$

for all $a \in (0, I_{E_m}(\mathrm{Vol}(E_m))$.

It is well known that $I_{E_m}^{-1}$ is absolutely continuous. In particular $(I_{E_m}^{-1})'$ exists almost everywhere.
Thus, since $I_{E_m}^{-1}(0) = 0 = I_{\mathbf{H}^n}^{-1}(0)$, to prove that $I_{E^m}^{-1} \leqslant I_{\mathbf{H}^n}^{-1}$ it suffices to show that

$$\boxed{(I_{E^m}^{-1})' \leqslant (I_{\mathbf{H}^n}^{-1})'}$$

But

$$\begin{aligned}(I_{E^m}^{-1})'(a) &= \big((n-1)H_\Sigma(a)\big)^{-1}\\ (I_{\mathbf{H}^n}^{-1})'(a) &= \big((n-1)H_0(a)\big)^{-1}\end{aligned}$$

where

$H_\Sigma(a)$ is the mean curvature of the hypersurface $\Sigma$ corresponding to *any* region $D \subset E_m$ with $\text{Area}(\Sigma) = a$ which has the largest volume.

$H_0(a)$ is the mean curvature of a hemisphere of area $a$.

So, to prove the relative isoperimetric inequality, all we need is to show that

$$\boxed{H_\Sigma(a) \geqslant H_0(a)}.$$

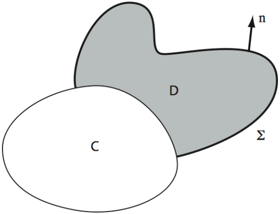
for all $a \in (0, I_{E_m}^{-1}(\mathrm{Vol}(E_m)))$.

That is, the mean curvature of any hypersurface $\Sigma$, trapping the smallest possible volume $v$ against $C$, is not greater than the mean curvature of a hemisphere with the same area as $\Sigma$.

We will also show that the equality holds only when $\Sigma$ itself is a hemisphere.

From now on let us fix $a$ and set $H_\Sigma := H_\Sigma(a)$, $H_0 := H_0(a)$.

To prove that $H_\Sigma \geqslant H_0$, let $\Sigma$ be oriented by the unit normals which point outside of $D$:




Let $k_i$, $i = 1, \ldots, n-1$, be the principal curvatures of $\Sigma$ with respect to $n$ and set

$$\Sigma^+ := \{p \in \Sigma \mid k_i(p) > 0\}.$$

Since $H$ is constant and positive,

$$\text{Area}(\Sigma) H_\Sigma^{n-1} = \int_\Sigma H_\Sigma^{n-1} \geqslant \int_{\Sigma^+} H_\Sigma^{n-1} \geqslant \int_{\Sigma^+} K,$$

the last inequality follows from the arithmetic vs. geometric mean inequalities.

So it remains to show that

$$\int_{\Sigma^+} K \geqslant \text{Area}(\Sigma) H_0^{n-1}$$

where, recall that, $H_0$ is the mean curvature of a hemisphere of the same area as $\Sigma$.

But if $H_0$ is the mean curvature of a hemisphere of the same area as $\Sigma$, then if $r$ denotes the radius of that hemisphere, we have

$$\begin{aligned}\text{Area}(\Sigma)H_0^{n-1} &= \left(\frac{\text{Area}(\mathbf{S}^{n-1})}{2}r^{n-1}\right)\left(\frac{1}{r}\right)^{n-1} \\ &= \frac{\text{Area}(\mathbf{S}^{n-1})}{2}.\end{aligned}$$

So, to prove that $\int_{\Sigma^+} K \geqslant \text{Area}(\Sigma)H_0^{n-1}$, it suffices to show that

$$\boxed{\int_{\Sigma^+} K \geqslant \frac{\text{Area}(\mathbf{S}^{n-1})}{2}}$$

This is all we need to prove the relative isop. ineq.

Further we will show that if equality holds in the above inequality, then $\partial\Sigma$ lies in a hyperplane, which takes care of uniqueness.

### Theorem

*Let $\Sigma \subset \mathbf{R}^n$ be a compact embedded hypersurface which is $C^{1,1}$ almost everywhere.*

*Suppose that $\partial\Sigma$ lies on the boundary of a convex $C^1$ set $C \subset \mathbf{R}^n$ and at every regular point $p \in \partial\Sigma$, the inward normal $\sigma(p)$ of $\Sigma$ is an outward normal of $C$.*

*Then*

$$\int_{\Sigma^+} |K| \geqslant \frac{\mathrm{Area}(\mathbf{S}^{n-1})}{2},$$

*and equality holds if and only if $\partial\Sigma$ lies in a hyperplane.*

Proving that $\int_{\Sigma^+} |K| \geqslant \text{Area}(\mathbf{S}^{n-1})/2$ is easy to do in two special cases:

**Easy Case 1:** $n = 3$ **and** $\chi(M) = 1$
Since $C$ is convex and $\Sigma$ lies outside of $C$ near $\partial\Sigma$, it follows that the geodesic curvature of $\partial\Sigma$ is nonpostitive everywhere. So

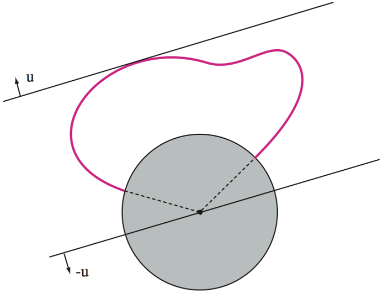
$$\int_{\partial\Sigma} \kappa_g \leqslant 0.$$

Consequently, by Gauss-Bonnet theorem:

$$\int_{\Sigma^+} K \geqslant \int_{\Sigma} K = 2\pi\chi(\Sigma) - \int_{\partial\Sigma} \kappa_g \geqslant 2\pi.$$

**Easy Case 2:** $C = \mathbf{S}^{n-1}$

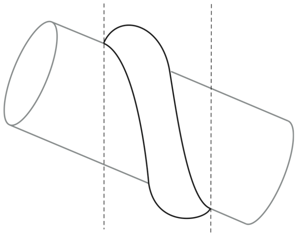
Connect all points of $\partial\Sigma$ by straight lines to the center $o$ of $\mathbf{S}^{n-1}$.




Then we obtain a closed surface $\widetilde{\Sigma}$. For every $u \in \mathbf{S}^{n-1}$, $\widetilde{\Sigma}$ must have a support hyperplane with outward normal $u$ or $-u$ which does not contain $o$ and is consequently tangent to $\Sigma$.

## Note

In general, however, it is not true that for every $u \in \mathbf{S}^{n-1}$, $\Sigma$ must have a tangent support hyperplane with outward normal $u$ or $-u$:

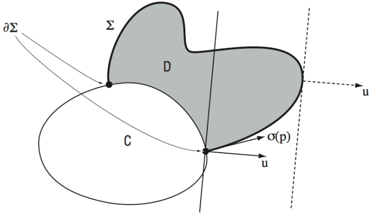


This is one of the reasons that proving $\int_{\Sigma^+} |K| \geqslant \text{Area}(\mathbf{S}^{n-1})/2$ is not so trivial.

Still, we will show that for at least half of the directions $u \in \mathbf{S}^{n-1}$, $\Sigma$ has a tangent support hyperplane with outward normal $u$.

Showing that the outward normals of tangent support hyperplanes of $\Sigma$ fill up at least half of the sphere is based on the following simple observation:

Let $p \in \partial\Sigma$, $\sigma(p)$ be the inward normal to $\partial\Sigma$ and $u \in \mathbf{S}^{n-1}$ be the outward normal to a support hyperplane of $\partial\Sigma$ such that $\langle \sigma(p), u \rangle > 0$.




Then $\partial\Sigma$ lies on one side of the hyperplane $H$ which passes through $p$ and is orthogonal to $u$, while some interior points of $\Sigma$ lie on the other side of $H$ where $\sigma(p)$ points.

So moving $H$ parallel to itself in the direction of $u$ we obtain a tangent support hyperplane with outward normal $u$.

So all we need to show is that the measure in $\mathbf{S}^{n-1}$ of all unit vectors $u$ such that

1. $\langle u, \sigma(p) \rangle > 0$, for some $p \in \partial\Sigma$
2. $u$ supports $\partial\Sigma$

adds up to at least half the area of $\mathbf{S}^{n-1}$.

In other words, let $N_p(\partial\Sigma)$ be the *support cone* of $\partial\Sigma$ at $p$, i.e.,

$$N_p(\partial\Sigma) := \{u \in \mathbf{S}^{n-1} \mid \langle q-p, u\rangle \leqslant 0, \text{for all } q \in \partial\Sigma\}.$$

This is just the set of unit vectors such that there exists a hyperplane through $p$ and with outward normal $u$ with respect to which $\partial\Sigma$ lies one side (the side opposite to $u$).

Then let

$$N_p(\partial\Sigma)/\sigma(p) := \{u \in N_p(\partial\Sigma) \mid \langle u, \sigma(p)\rangle > 0\}.$$

be the support cone modified by $\sigma$

and set

$$N(\Sigma)/\sigma := \bigcup_{p\in\partial\Sigma} N_p(\partial\Sigma)/\sigma(p).$$

What we need to show is that

$$\boxed{\mu\big(N(\Sigma)/\sigma\big) \geqslant \frac{1}{2}\,\text{Area}(\mathbf{S}^{n-1})}$$

where $\mu$ denotes the measure in $\mathbf{S}^{n-1}$.

What we need is a special case of the following general fact:

### Theorem

*Let $X \subset \mathbf{R}^n$ be a compact set which is disjoint from the relative interior of its convex hull (e.g., let $X = \partial\Sigma$).*

*Suppose there exists a continuous mapping $\sigma\colon X \to \mathbf{S}^{n-1}$ such that $\sigma(p) \in N_pX$ for all $p \in X$ (e.g. $\sigma$ is the inward normal of $\partial\Sigma$).*

*Then*

$$\mathrm{Area}\,(NX/\sigma) \geqslant \frac{\mathrm{Area}(\mathbf{S}^{n-1})}{2}.$$

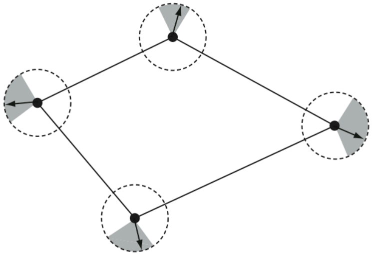
*Equality holds if and only if $X$ lies in a hyperplane $\Pi$, and $\sigma(p)$ is orthogonal to $\Pi$ for all exposed points $p \in X$.*

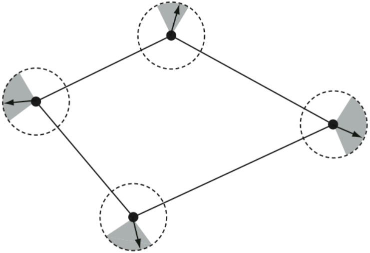
To see why the last claim is true, first suppose that $X$ is a convex polytope, or is discrete (e.g. $\partial\Sigma$ is polygonal). Then

$$N(X) = \sum_i N_{p_i}(X)$$

where $p_i$ are the vertices of $X$, and

$$N(X)/\sigma = \sum_i N_{p_i}(X)/\sigma_i$$





Note that, by convexity,

$$\sum_i \mu(N_{p_i}(X)) = \mu(N(X)) = \text{Area}(\mathbf{S}^{n-1}).$$

So when $X$ is dicsrete to prove that $\mu(N(X)/\sigma) \geqslant \text{Area}(\mathbf{S}^{n-1})/2$ all we need is to show that

$$\boxed{\mu(N_{p_i}(X)/\sigma_i) \geqslant \frac{1}{2}\mu(N_{p_i}(X))}$$

But

$$N_p(X)/\sigma = N_p(X) \cap H_\sigma$$

where $H_\sigma$ is a hemisphere centered at $\sigma$.

Further, $N_p(X)$ is a convex subset of $\mathbf{S}^{n-1}$.

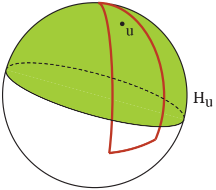
So, when $X$ is discrete, all we need to show is that ...

## Lemma

*Let $A \subset \mathbf{S}^{n-1}$ be a (geodesically) convex set, and $u \in A$. Then*

$$\mu(A \cap H_u) \geqslant \frac{1}{2}\mu(A).$$

*where $H_u$ is a hemisphere centered at $u$. Furthermore equality holds if and only if $-u \in A$.*




This is the deep reason behind the relative (and the classical!) isoperimetric inequality.

To prove that Area $\left(NX/\sigma\right) \geqslant \frac{\text{Area}(S^{n-1})}{2}$ for general sets, we approximate them by polytopes and apply the previous lemma.

The strict inequality then will get pushed through the limit easily.

In order to push through the sharp inequality, we need a stronger version of the previous lemma which includes a *stability result* as well ...

### Lemma

*Let $A \subset \mathbf{S}^{n-1}$ be a (geodesically) convex set, and $u \in A$. Then*

$$\mu(A \cap H_u) \geqslant \frac{1}{2}\mu(A).$$

*where $H_u$ is a hemisphere centered at $u$. Equality holds if and only if $-u \in A$.*
*Further, for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\mu(X \cap \mathbf{H}_u) \leqslant \left(\frac{1}{2} + \delta\right)\mu(X) \implies \operatorname{dist}(X, -u) \leqslant \epsilon.$$

## Generalization: The Capillary Isopermetric Inequality

Let $C$ be a convex domain in $\mathbf{R}^n$ and $D$ be a domain in $\mathbf{R}^n - C$. Define the capillary volume

$$Vol_C^\theta(\partial D) := Vol(\partial D - \partial C) - (\cos\theta) Vol(\partial D \cap \partial C)$$

where $0 \leqslant \theta \leqslant \pi$. Then

$$Vol_C^\theta(\partial D) \geqslant Vol_{\mathbf{H}^n}^\theta(\partial B)$$

where $\mathbf{H}^n$ is a half space of $\mathbf{R}^n$ and $B^n \subset \mathbf{R}^n$ is a ball such that $B := B^n \cap \mathbf{H}^n$ has corner angle $\theta$ and $Vol(D) = \mathrm{Vol}(B)$. Equality holds if and only if $D = B$.

Thank you!