Total Absolute Curvature of Surfaces in Cartan-Hadamard Manifolds

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Joint work with Hoisington, Raffaelli, Stavroulakis

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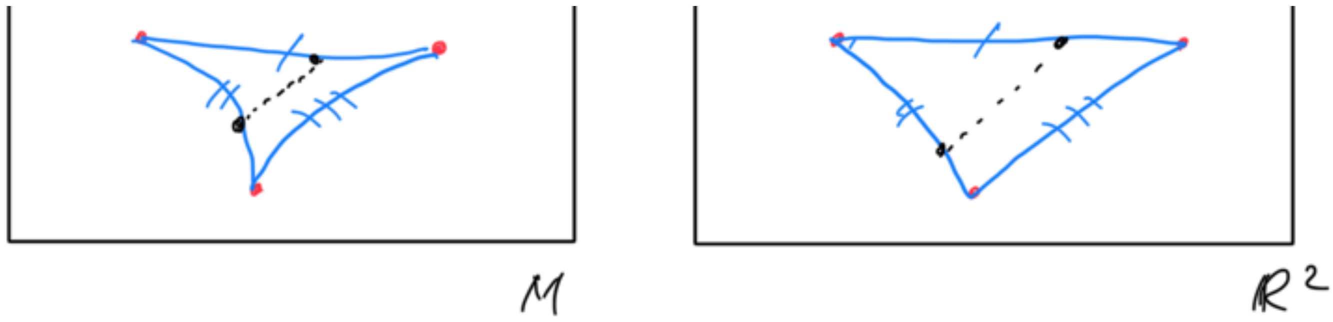
- A Cartan-Hadamard manifold is a complete simply connected Riemannian manifold with sectional curvature $K \leq 0$.

Example: $\mathbb{R}^n$ & $\mathbb{H}^n$

Basic Properties:

* M^n is diffeomorphic to $\mathbb{R}^n$
* Unique geodesic between every pair of points
* CH-mflds are CAT(0) spaces

geodesic metric spaces where triangles are thinner than those in $\mathbb{R}^2$.



* Convex sets are natural.

  sets which contain the geodesic between every pair of points.

- So basically CH-mflds generalize $\mathbb{R}^n$ & H^n at the expense of losing both linearity & homogeneity.

Main Problems:

* The isoperimetric inequality (Cartan-Hadamard conjecture)

* The total curvature inequality.

Total Curve Ineq. $\Rightarrow$ Isop. Inequality

Let $\Gamma \subset M^n$ be a convex hypersurfaces

boundary of a convex body

Let G.K be the Gauss-Kronecker curvature

det of the 2nd fund. form.

The <u>total curvature</u> is defined as

$$\boxed{\mathcal{G}(\Gamma) := \int_\Gamma GK}$$

The total curvature inequality states that

$$\mathcal{G}(\Gamma) \geq |S^{n-1}|.$$

- The Isop. inequality is known only for
  $n=2$, $n=3$, $n=4$.
  (A. Weil, 1920s) (Kleiner 1990s) (Croke 1980s)
- The total curvature ineq. is known only for $n=2$, $n=3$, as immediate consequences of Gauss-Bonnet thm & Gauss' equation.
- Both ineq. holds in H^n & for spheres in any CH-mfld.
- Recent results (Joint work with John Stavroulakis):
  - ☆ The isop. ineq. is locally true, that is perturbing the metric of the unit ball $B^n \subset \mathbb{R}^{n+1}$ to a metric with nonpositive curvature does not reduce the isop. ratio.

IMRN, 2026.

* The total curvature inequality holds if K is const near Γ. (Joint with Stavroulakis) arXiv, 2026

Chern-Lashof Thm

For any closed hypersurface $\Gamma \subset \mathbb{R}^n$ let

$$\tilde{\mathcal{G}}(\Gamma) := \int_\Gamma |GK|$$

be the <u>total absolute curvature</u>.

Chern-Lashof showed that

$$\tilde{\mathcal{G}}(\Gamma) \geq |S^{n-1}|$$

with "=" $\Longleftrightarrow$ Γ is convex.

Main Result of this talk:

Thm [——, Heisington, Raffaelli, Stavroulakis]

For any closed surface $\Gamma \subset M^3$,

$$\tilde{\mathcal{G}}(\Gamma) \geq 4\pi$$

with "=" $\iff$ Γ bounds a flat convex body.

$K=0$

– This had been given as an exercise by Gromov in the 1985 book with Ballmann & Schroeder on Spaces of nonpositive curvature.

This talk is about proving the above result.

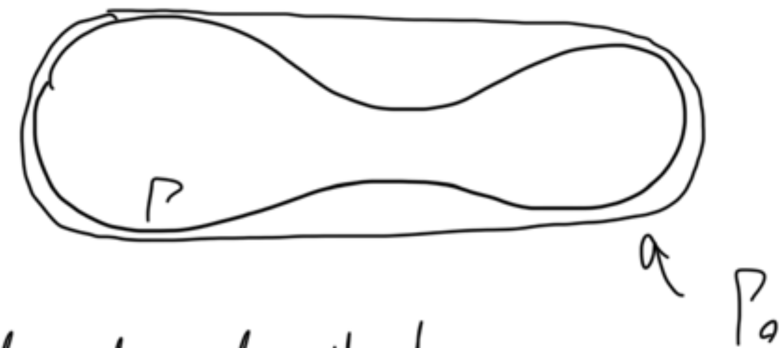
Some history:

- Schroeder & Strake proved the thm for strictly convex Γ in 1989.
- G. & Spruck refined Schroeder & Strake in 2022 to spaces with curvature bounded above by $k \leq 0$.
- Petrunin described a proof for rigidity of smooth hypersurfaces $\Gamma \subset M^n$ with $K_M(T\Gamma)=0$, which applies to the simply connected case ($\pi_1(\Gamma)=0$). 2025

Proof.

Step 0': The Equality

Let $\Gamma_0 := \partial\, \mathrm{conv}(\Gamma)$



Kleiner had observed that
(see G.-Spruck 2022)

$$\tilde{\mathcal{G}}(\Gamma) \geq \tilde{\mathcal{G}}(\Gamma_0) = \mathcal{G}(\Gamma_0)$$

Here

$$\mathcal{G}(\Gamma_0) := \lim_{t \to 0} \mathcal{G}(\Gamma_0^t)$$

outer parallel surfaces
which are $C^{1,1}$

By Gauss' Equation:

$$GK_{\Gamma_0^t}(p) = K_{\Gamma_0^t}(p) - K(T_p \Gamma_0^t)$$

intrinsic   extrinsic

Integrating we obtain:

$$\mathcal{G}(\Gamma_0^t) = 4\pi - \int_{p \in \Gamma_0^t} K(T_p \Gamma_0^t) \geq 4\pi$$

So the inequality is obtained quickly.

Next we prove the equality case.

Step 1: Regularity of the convex hull:

Γ_0 is C^1.

Uses a result of Borbely.

& Basic theory of tangent cones

& semiconvex functions.

Step 2: If "=" holds $\left(\tilde{\mathcal{G}}(\Gamma) = 4\pi\right)$, then

$$K(T_p\Gamma_0) = 0, \quad \forall p \in \Gamma_0$$

We have:

$$\tilde{\mathcal{G}}(\Gamma) \geq \mathcal{G}(\Gamma_0) \geq 4\pi$$

So if $\tilde{\mathcal{G}}(\Gamma) = 4\pi \Rightarrow \mathcal{G}(\Gamma_0) = 4\pi$

$$4\pi = \mathcal{G}(\Gamma_0) = \lim_{t\to 0} \mathcal{G}(\Gamma_0^t) = 4\pi - \lim_{t\to 0}\int_{\Gamma_0^t} K(T_p\Gamma_0^t)$$

$$\Rightarrow K(T_p\Gamma_0) = 0$$

Step 3: It remains to show:

Thm: If $\Gamma \subset M^3$ is a C^1 convex surface with $K(\Gamma) = 0$

then $K \equiv 0$ on the convex body

bounded by Γ.

This theorem is proved as follows:

3.1: Construct parallel frame along Γ.

(Ambrose-Singer thm)

3.2: Isometrically embed Γ into $\mathbb{R}^3$ concave while preserving the total curvature of all C^1 curves in Γ.

(Pogorelov's theory of surfaces with bounded extrinsic curvature)

3.2: Extend the isometric embedding to the convex body bounded by Γ.

(A generalization of Schur caparison thm, Reshetnyak majorization, Kirzsbraun-Lang-Schroeder)

Step 3.1: The Parallel Frame

Since $K \leq 0$ & $K(T\Gamma) \equiv 0$,

$$R_M(X,Y)Z = 0, \quad X,Y \in T\Gamma, \; Z \in TM$$

So parallel transport in $TM|_\Gamma$ is path independent.

Pick an orthonormal frame e_1, e_2, e_3 at some point $p_0 \in \Gamma$ and parallel translate it all over Γ.

Step 3.2: The isometric embedding

3.2.1 Construction of the embedding

Let $\theta_i := \langle \cdot , e_i \rangle$ be the dual 1-forms.

For any $p \in \Gamma$, let γ be a C^1 curve from p_0 to p. Set

$$f_i(p) := \int_\gamma \theta_i$$

Then $f = (f_1, f_2, f_3) : \Gamma \longrightarrow \mathbb{R}^3$ is a

C^1 isometric embedding. Indeed

$$d\theta_i(X,Y) := X(\theta_i(Y)) - Y(\theta_i(X)) - \theta_i([X,Y])$$

$$= X\langle Y, e_i\rangle - Y\langle X, e_i\rangle - \langle [X,Y], e_i\rangle$$

$$= \langle D_X Y, e_i\rangle + \langle Y, D_X e_i\rangle - \langle D_Y X, e_i\rangle - \langle X, D_Y e_i\rangle - \langle [X,Y], e_i\rangle$$

$$= \langle D_X Y - D_Y X - [X,Y], e_i\rangle$$

$$= 0$$

So f is independent of the choice of γ.

Furthermore, by Poincare lemma,

$$df_i = \theta_i$$

So

$$\langle df(X), df(Y)\rangle = df_i(X)df_i(Y) = \theta_i(X)\theta_i(Y) = \langle X, Y\rangle$$

Thus f is an isometric embedding

3.2.2 Convexity

C^1 isometric embedings are very flexible by the Nash-Kuiper theory. So it is not automatic the f is convex. But:

- Intrinsic curvature measure in the sense of Alexandrov-Zalgaller is ≥ 0.
  (In terms of excess of triangles)

- The extrinsic curvature is bounded in the sense of Pogorelov
  (the Gauss map has bounded variation)

- By Pogorelov's generalization of ~~Gauss' Theorema Egregium~~

  Intrinsic curvature = Extrinsic curvature
  (for surfaces of bounded extrinsic curvature)

- So $f(\Gamma)$ has nonnegative ext. curvature $\Rightarrow$ $f(\Gamma)$ is convex

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Step 3.3: Extension of the isometric embedding to the entire convex hull

For any C^1 unit-speed curve $\gamma: [a, \ell] \to M$ we define the <u>total curvature</u> $\tau(\gamma)$ as follows. Parallel transport $\gamma'(t)$ along

γ to $\gamma(0)$ to obtain a curve in the unit sphere $S_{\gamma(0)}M \subset T_{\gamma(0)}M$.

The $\tau(\gamma)$ is the length of that curve.

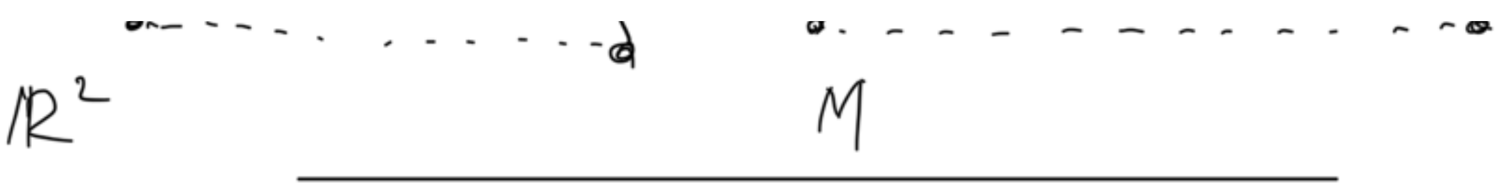
The isometric embedding $f: \Gamma \to \mathbb{R}^3$ constructed above preserves the total curvature of all C^1 curves.

Thm [Generalization of Schur's bow lemma to CH-mflds]

Let $\gamma_1 : [0,\ell] \to \mathbb{R}^2$, $\gamma_2 : [0,\ell] \to M^n$ be C^1 unit speed curves. Suppose that γ_1 is chord convex, & for every interval $I \subset [0,\ell]$, $\tau(\gamma_2(I)) \leq \tau(\gamma_1(I))$.

Then

$$|\gamma_2(0)\gamma_2(\ell)| \geq |\gamma_1(0)\gamma_1(\ell)|$$

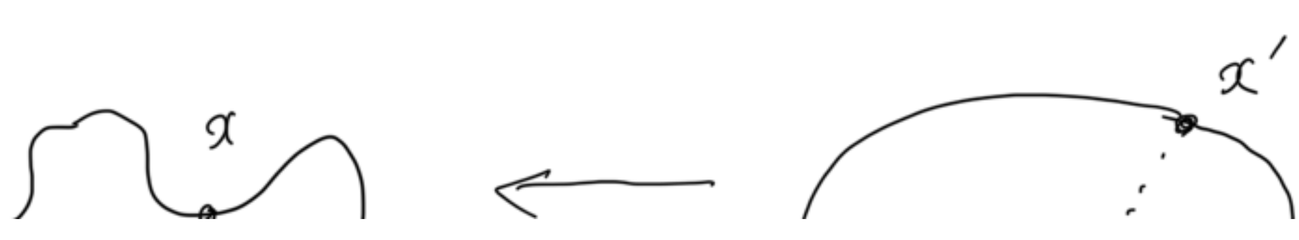


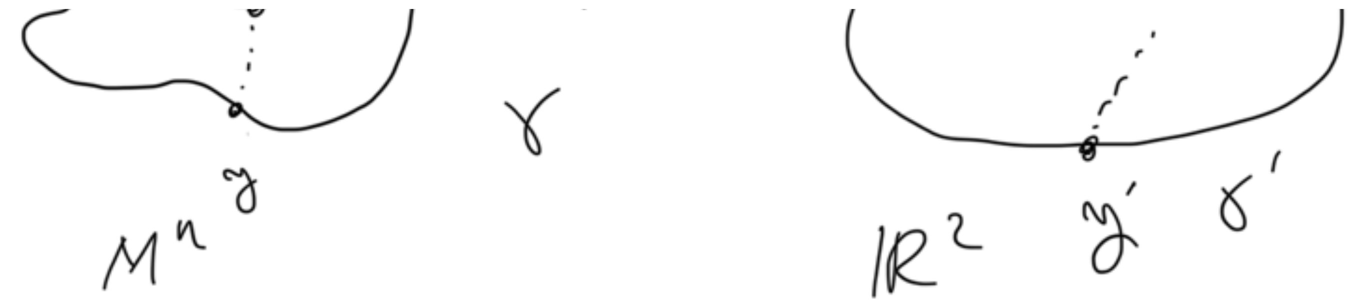
To complete the solution to Gromov's problem it remains to show:

Proposition: Let $C \subset M^n$, $C' \subset \mathbb{R}^n$ be convex bodies with C^1 boundaries Γ, Γ'. Suppose there exist a C^1 isometry $f : \Gamma \to \Gamma'$ which preserves the total curvature of C^1 curves in Γ. Then f extends to an isometry $C \to C'$.

We need the notion of majorization.

A convex curve $\gamma' \subset \mathbb{R}^2$ majorizes a closed curve $\gamma \subset M^n$ if there exists an arclength preserving map $\gamma' \to \gamma$ which does not reduce chord-lengths:



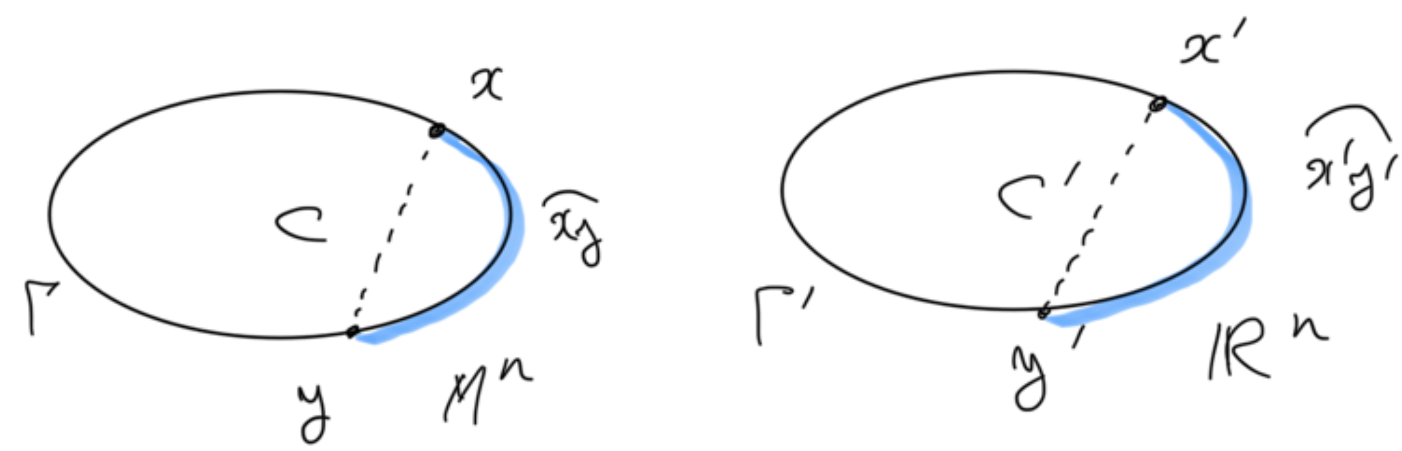


$$|xy| \leqslant |x'y'|$$

Reshetnyak showed that γ' exists.

Now we prove the proposition.

Proof:



For any $x \in \Gamma$, let $x' := f(x)$.

- It suffices to show that $|\widehat{xy}| = |\widehat{x'y'}|$ by Reshetnyak majorization & Kirszbraun-Lang-Schroeder extension.

- Let $\widehat{x'y'}$ be an arc of the curve $\gamma' := \Gamma' \cap \Pi$ for a plane Π passing through x' & y'

Set $\widehat{xy} := f^{-1}(\widehat{x'y'})$. Then

$$\tau(\widehat{xy}) = \tau(x'y')$$

$\Longrightarrow$ by Schur's bow Lemma $\quad |xy| \geq |x'y'|$

- By Reshetnyak $\exists$ a convex curve $\gamma'' \subset \Pi$ which majorizes γ. It follows from the above inequality that γ'' majorizes γ'.

- Since γ'' & γ' are both convex planar curves they must be congruent. So γ' majorizes γ:

$$|xy| \leq |x'y'|.$$

How is the generalized Schur bow lemma proved?

$$\tau(\gamma_2) \leq \tau(\gamma_1) \Rightarrow |\gamma_2(0)\gamma_2(\ell)| \geq |\gamma_1(0)\gamma_2(\ell)|$$

It is enough to show:

Prop. <u>For every C^1 curve $\gamma: [0,\ell] \to M$ there exist a chord-convex C^1 proper majorization $\tilde{\gamma}: [0,\ell] \to \mathbb{R}^2$ which does not increase curvature.</u>

- If $\tilde{\gamma}$ is $C^{1,1}$, this is automatic by Reshetnyak due to the estimate:

$$|\gamma(t-h)\gamma(t+h)| = 2h - \frac{\kappa(t)^2}{3} h^3 + o(h^3)$$

- We pass on to the C^1 case by an approximation argument, which involves applying Helly's selection thm for monotone functions to turning angle of planar curves.

One remaining question (among many)

Chern-Lashof thm implies that a closed surface with $GK \geq 0$ in $\mathbb{R}^3$ is convex. This is a generalization of of 1898 Thm of Hadamard for $GK > 0$.

But this fact is not known in M^3.

Stephanie Alexander proved this for $GK > 0$ in M^3.

How was the simply connected case proved in the previous paper based on Petrunin?

Suppose $\Gamma \subset M^3$ is simply connected.

& $\tilde{\mathcal{G}}(\Gamma) = 4\pi$. Then

Recall that by Gauss' Equation

$$GK(p) = K_\Gamma(p) - K_M(T_p\Gamma) \geq K_\Gamma(p)$$

By Gauss-Bonnet we also have

$$\mathcal{G}(\Gamma) = \int_\Gamma K_\Gamma = 4\pi$$

So

$$\tilde{\mathcal{G}}(\Gamma) \geq \mathcal{G}(\Gamma) \geq 4\pi$$

$$\Rightarrow \quad \tilde{\mathcal{G}}(\Gamma) = \mathcal{G}(\Gamma)$$

$$\Rightarrow \quad GK \geq 0$$

Recall that we also know that

$$K_M(T\Gamma) = 0$$

It follows from the isometric embedding, constructed by the Gauss-Codazzi equations, that Γ is convex.

More details about the proof of regularity of the convex hull (Γ is $C^1 \Rightarrow \Gamma_0 := \partial\,\mathrm{Conv}(\Gamma)$ is C^1.)

Enough to show that tangent cones of

Γ ... all ...

Γ_0 are JHM.

- Need to check this only at points $p \in \Gamma_0 - \Gamma$.
- There are two cases:

  (i) p lies on a geodesic with endpoints q_1, q_2 on Γ.

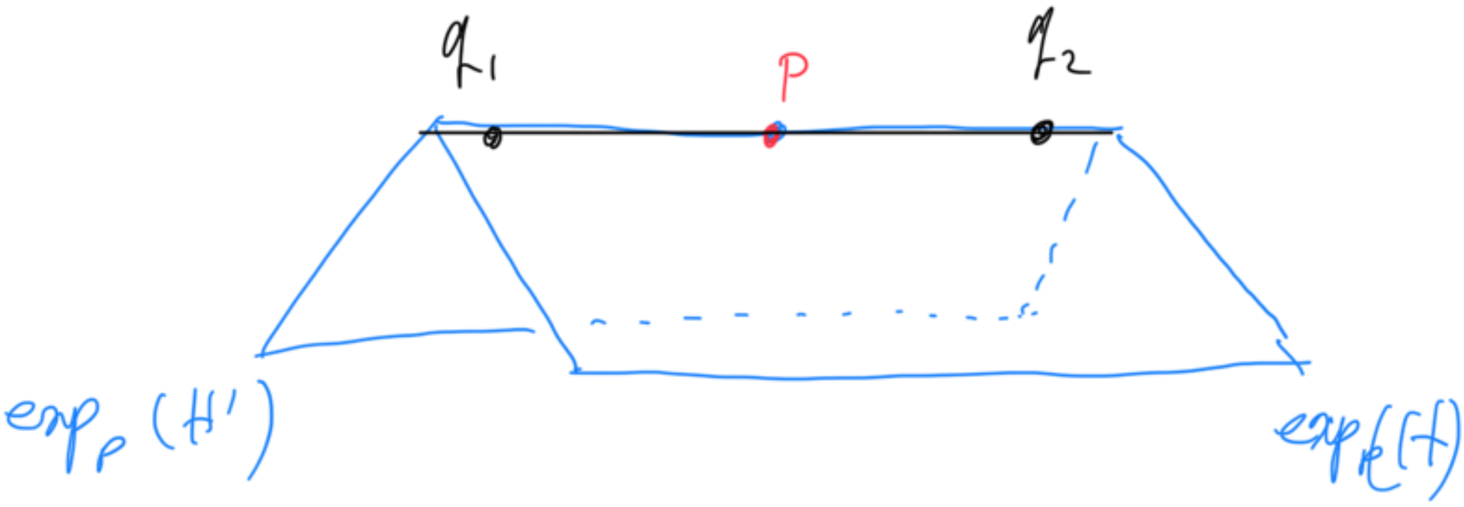
  (ii) Not.

Case (ii) was covered by Barbelly.

Suppose Case (i).

If $T_p \Gamma_0$ is not flat, then $\exp_p^{-1}(\Gamma_t)$ is supported at p by distinct support planes, H, H'. Then $\exp_p(H)$, $\exp_p(H')$ are complete surfaces in M^3 supporting Γ_0 at p & containing the geodesic. But these surfaces are transversal. So Γ

cannot be C^1 near q_1 & q_2.



More details on the proof of the generalized Schur theorem:

We make various estimates via expansions in normal coordinates:

Lemma. Choose normal coordinates centered at a point $o \in M^n$. For $a, b \in T_o M \simeq \mathbb{R}^n$,

$$|\exp_o(a)\exp_o(b)| = |ab|^2 - \frac{1}{3}R(a,b,b,a) + O((|a|+|b|)^5)$$

Lemma. Let $\gamma : [0,\ell) \to M^n$ be a C^1 unit speed curve, and $t \in (0,\ell)$ be a twice differentiable point. Then

$$|\gamma(t-h)\gamma(t+h)| = 2h - \frac{\kappa(t)^2}{3}h^3 + o(h^3)$$

Lemma. If γ is $C^{1,1}$, we have the uniform estimate

$$|\gamma(t-h)\gamma(t+h)| \geq 2h - Ch^3$$

These estimates allow us to prove the result in the $C^{1,1}$ case & we pass to the C^1 case by approximation.