Four-Vertex Theorems in Riemannian Surfaces

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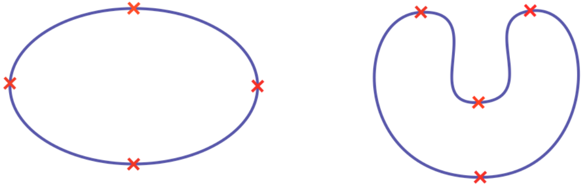
Theorem (Kneser,1912)

Any simple closed curve in $\mathbf{R}^2$ *has (at least) four vertices (local extrema of curvature)*



How about nonsimple curves?

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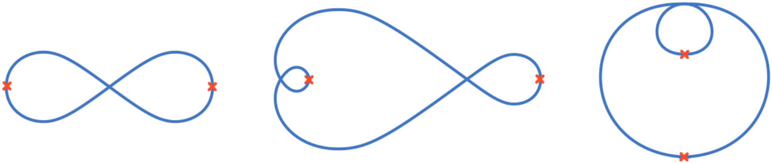

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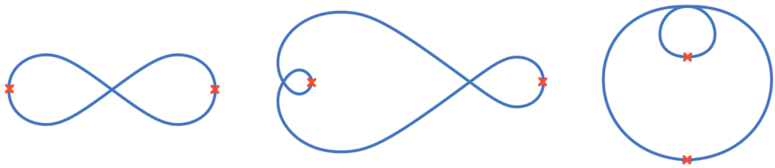

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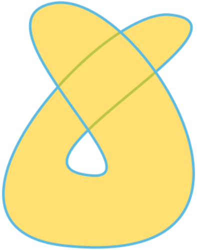

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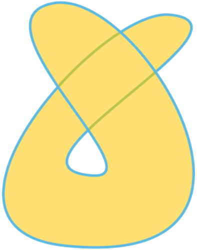

The same result also holds in $\mathbf{S}^2$ and $\mathbf{H}^2$, because the stereographic projection $\pi\colon \mathbf{S}^2 - \{0,0,1\} \to \mathbf{R}^2$ and the inclusion map $i\colon \mathbf{H}^2 \to \mathbf{R}^2$ preserve vertices.

Could Pinkall's theorem be a hint of a purely intrinsic or Riemannian version of the four vertex theorem?

More precisely:

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Let M be a compact surface with boundary and constant curvature. Must the boundary of M have 4-vertices (in terms of geodesic curvature)?

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Theorem

Let M be a compact surface with boundary ∂M. Then every metric of constant curvature induces four vertices on ∂M if and only if M is simply connected.

Indeed, when M is not simply connected, there are elliptic, parabolic and hyperbolic metrics of constant curvature on M which induce only two vertices on ∂M.

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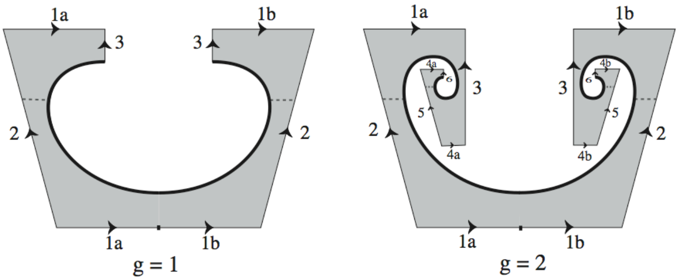
1a
1b
3
3
2
2
1a
1b
g = 1
1a
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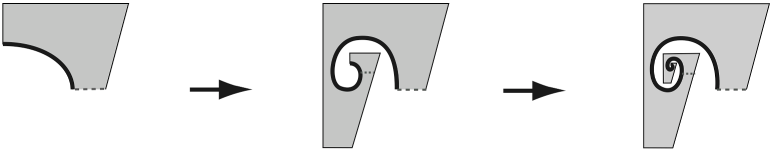
4b
1a
1b
g = 2

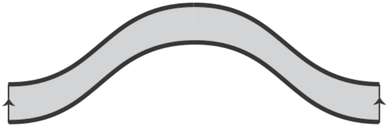


Flat metrics with fewest vertices

First we show that if M is not simply connected, it admits a flat metric with only two vertices on each boundary component.

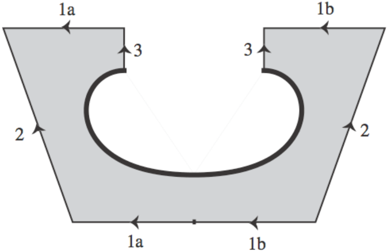
Recall that M is homeomorphic to a closed surface $\overline{M}$ minus k-disks. There are three special cases that we consider first:

I. $\overline{M} = \mathbf{S}^2$ & $k = 2$



II. $\overline{M} = \mathbf{RP}^2$ & $k = 1$

III. $g(\overline{M}) = 1$ & $k = 1$



$$\kappa(t) = 1 - \frac{3}{4}\cos(t),$$

where $-\pi \leq t \leq \pi$. More explicitly, $\gamma(t) := \int_0^t e^{i\theta(s)}ds$, where $e^{i\theta} := (\cos(\theta), \sin(\theta))$, and $\theta(t) := \int_0^t \kappa(s)ds$.

In all the remaining cases we will show that $\overline{M}$ admits a flat metric with exactly k conical singularities.

Then we remove these singularities by cutting $\overline{M}$ along simple closed curves which have only two critical points of geodesic curvature each.

If $\overline{M}$ has k singularities of angles θ_i, then by Gauss-Bonnet theorem,

$$\sum_{i=1}^{k}(2\pi - \theta_i) = 2\pi\chi(\overline{M}).$$

Troyanov has shown that the above condition is also sufficient for the existence of flat metrics with conical singularities of prescribed angles. This quickly yields

Lemma

Suppose $k(\overline{M}) \geq 3, 2, 2, 1$, according to whether $\overline{M} = \mathbf{S}^2$, $\overline{M} = \mathbf{RP}^2$, $g(\overline{M}) = 1$, or $g(\overline{M}) \geq 2$ respectively. Then there exists a flat metric on $\overline{M}$ with exactly k conical singularities.

Lemma

Let C be a cone with angle $\phi \neq 2\pi$ and Γ be a circle centered at the vertex of C. Then there exists a C^∞ perturbation of Γ which has only two critical points of curvature.

Proof.

If $\phi = 2n\pi$ (where $n \geq 2$), let

$$r_\lambda(\theta) := 1 - \lambda \cos\left(\frac{\theta}{n}\right).$$

If $\phi \neq 2n\pi$, we cut a segment of theses curves. □

Proposition

Let M be a compact surface with boundary and flat metric g_0. Then there exists a family g_λ of Riemannian metrics on M, $\lambda \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, such that g_λ has constant curvature λ, and $\lambda \mapsto g_\lambda$ is continuous with respect to the C^∞ topology.

This is easy when M is simply connected, for then it isometrically immersed into the plane and we may perturb the whole plane

$$(g_\lambda)_{ij}(x) := \frac{\delta_{ij}}{\left(1 + \frac{\lambda}{4}\|x\|^2\right)^2}.$$

But how about the general case:

Perturbations of Flat Metrics

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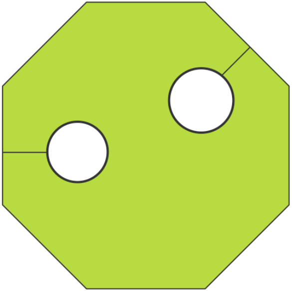

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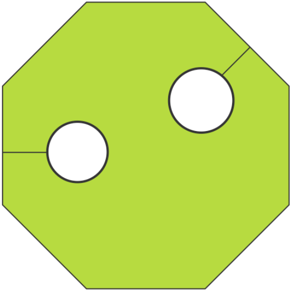

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Lemma (Fund. thm. of curves for Riemannian surfaces)

Let M be a complete oriented C^∞ Riemannian surface, $p \in M$, and $u \in T_pM$ be a unit vector. Suppose that we are given a C^∞ function $\kappa\colon I \to \mathbf{R}$, for some open interval $I \subset \mathbf{R}$ with $0 \in I$. Then there exists a unique unit speed C^∞ curve $\gamma\colon I \to M$ with $\gamma(0) = p$, $\gamma'(0) = u$, and geodesic curvature $\kappa(t)$.

Proof.

This follows from basic ODE theory:

$$g\Big(\nabla_{\gamma'(t)}\gamma', J\big(\gamma'(t)\big)\Big) = \kappa(t)$$

which may be rewritten as

$$\begin{aligned} v_1' &= -\sum_{i,j=1}^{2} v_i v_j \Gamma_{ij}^1(\gamma_1,\gamma_2) - \frac{\kappa}{2Gv_2}, \\ v_2' &= -\sum_{i,j=1}^{2} v_i v_j \Gamma_{ij}^2(\gamma_1,\gamma_2) + \frac{\kappa}{2Gv_1}. \end{aligned}$$

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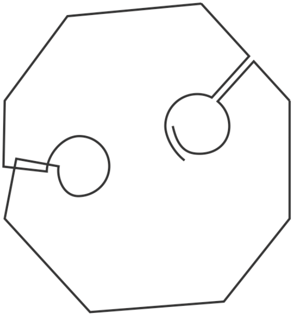

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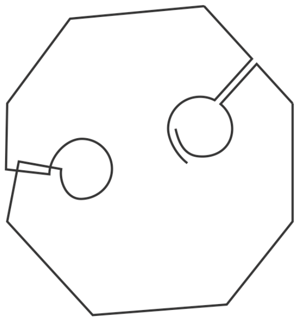

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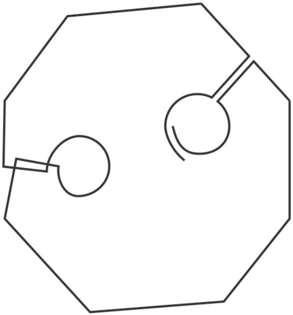

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Recall that by Kneser's theorem, and its extension to $\mathbf{H}^2$ and $\mathbf{S}^2$, *any simple closed curve in a simply connected space form has four vertices*.

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The only complete Riemannian surfaces M where every simple closed curve has more than two vertices are the the space forms with finite fundamental group (i.e., $\mathbf{R}^2$, $\mathbf{S}^2$, $\mathbf{H}^2$, and $\mathbf{RP}^2$, up to a rescaling).

Proof.

Suppose that M has Kneser's four vertex property.

Then by Jackson's theorem M has constant curvature $K = 1$, 0, or -1.

Then $M = X/G$, where $X = \mathbf{R}^2$, $\mathbf{S}^2$, or $\mathbf{H}^2$.

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By a theorem of Mobiüs, every simple closed noncontractible curve Γ in $\mathbf{RP}^2$ has at least three inflection points. So, it must have at least three vertices as well.

If, on the other hand, Γ is contractible, then it lifts to a pair of closed curves $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ in $\mathbf{S}^2$ and the covering is one-to-one on each of these curves.

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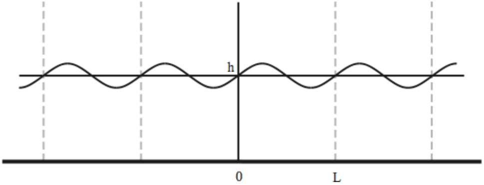

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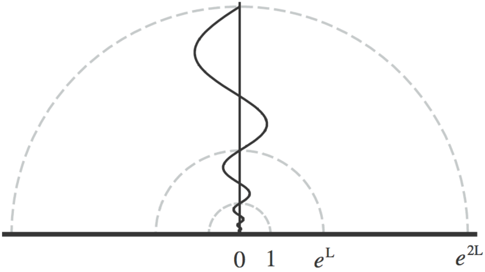

$$\left(\lambda t \sin\left(\frac{\pi}{L}\ln(t)\right), t\right).$$

Another four-vertex theorem for complete surfaces

Recall that by Pinkall's theorem, and its extension to $\mathbf{H}^2$ and $\mathbf{S}^2$, *any closed curve bounding a compact surface in a simply connected space form has four vertices.*

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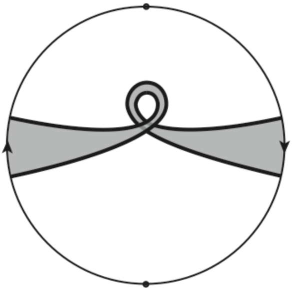
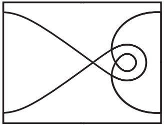


II: The parabolic case ($K = 0$)

So how does one construct a closed curve with only two vertices which bounds a compact immersed surface on a cylinder?

It is not so hard to construct one on a torus:



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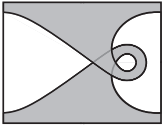

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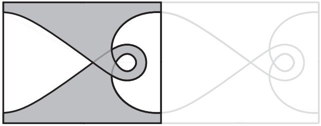

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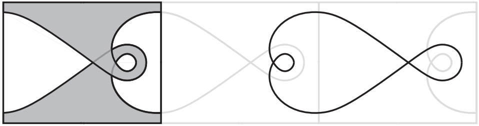

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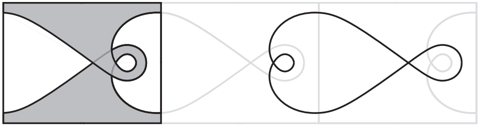
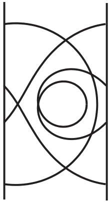
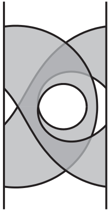
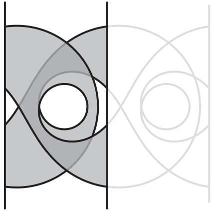
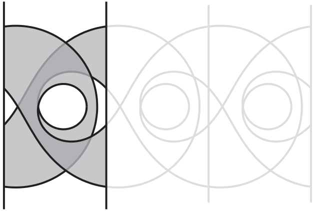


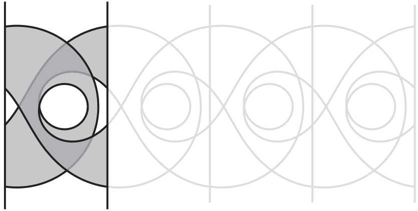
But for a cylinder this will be more complicated:

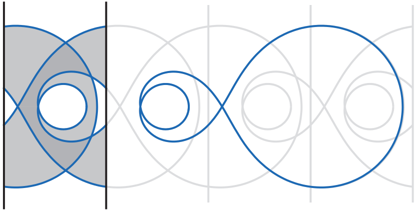


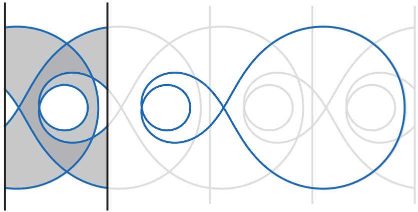




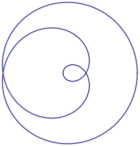




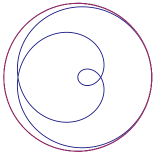




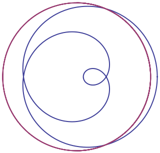
$$\frac{1}{a^2 + 2a\cos\left(\frac{t}{5}\right)\cos(t) + \cos\left(\frac{t}{5}\right)^2}\left(a + \cos\left(\frac{t}{5}\right)\cos(t),\ \cos\left(\frac{t}{5}\right)\sin(t)\right)$$



$$r(\theta) = \cos\left(\frac{\theta}{5}\right)$$

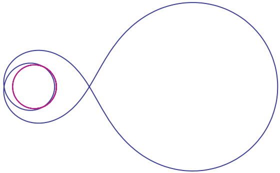


$$\left(\cos(\theta)\cos\left(\frac{\theta}{5}\right), \sin(\theta)\cos\left(\frac{\theta}{5}\right)\right)$$



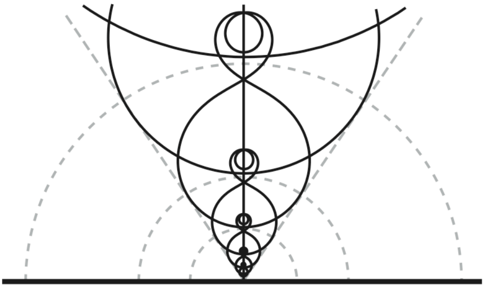
$$\left(0.9+\cos(\theta)\cos\left(\frac{\theta}{5}\right),\sin(\theta)\cos\left(\frac{\theta}{5}\right)\right)$$

$$\left(0.9 + \cos(\theta)\cos\left(\frac{\theta}{5}\right), \sin(\theta)\cos\left(\frac{\theta}{5}\right)\right)$$



$$\frac{1}{a^2 + 2a\cos\left(\frac{t}{5}\right)\cos(t) + \cos\left(\frac{t}{5}\right)^2}\left(a + \cos\left(\frac{t}{5}\right)\cos(t),\ \cos\left(\frac{t}{5}\right)\sin(t)\right)$$

III: The hyperbolic case ($K = -1$)



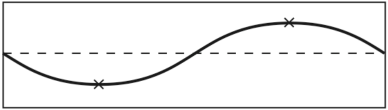
Theorem

The only complete Riemannian surfaces where every closed curve which bounds a compact embedded surface has more than two vertices are orientable space forms of genus zero, flat tori, and rescalings of $\mathbf{RP}^2$.

Theorem

Let Γ be a closed geodesic of length L in a Riemannian 2-manifold of constant curvature K, which is orientable near Γ.

Then, every neighborhood of Γ contains a closed curve which has only two vertices, and may be required to be arbitrarily C^∞-close to Γ, if, and only if, $K \neq (2\pi/L)^2$.



The End