TOTAL CURVATURE OF CONVEX HYPERSURFACES IN CARTAN-HADAMARD MANIFOLDS

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Abstract. We show that if the curvature of a Cartan-Hadamard n-manifold is constant near a convex hypersurface Γ, then the total Gauss-Kronecker curvature $\mathcal{G}(\Gamma)$ is not less than that of any convex hypersurface nested inside Γ. This extends Borbély's monotonicity theorem in hyperbolic space. It follows that $\mathcal{G}(\Gamma)$ is bounded below by the volume of the unit sphere in Euclidean space $\mathbf{R}^n$.

1. Introduction

A Cartan-Hadamard manifold M^n is a complete simply connected space with nonpositive curvature. A *convex hypersurface* $\Gamma \subset M$ is the boundary of a compact convex set with interior points. An outstanding question in Riemannian geometry [4, p. 66][22] is whether the total Gauss-Kronecker curvature

$$\mathcal{G}(\Gamma) \geq |\mathbf{S}^{n-1}|, \tag{1}$$

where $|\mathbf{S}^{n-1}|$ is the volume of the unit sphere in Euclidean space $\mathbf{R}^n$. Establishing this inequality would resolve the Cartan-Hadamard conjecture [15, 20] concerning the extension of the Euclidean isoperimetric inequality to manifolds of nonpositive curvature [2, 7, 19]. It is known that (1) holds for geodesic spheres [15], and Borbély [5] showed that it holds in hyperbolic space $\mathbf{H}^n$. More generally, he proved that if Γ, γ are convex hypersurfaces in $\mathbf{H}^n$ with γ nested inside Γ, then $\mathcal{G}(\Gamma) \geq \mathcal{G}(\gamma)$. We refine this monotonicity result as follows:

Theorem 1.1. *Let Γ, γ be convex hypersurfaces in a Cartan-Hadamard manifold M^n, with γ nested inside Γ. Suppose that the curvature K of M is constant on a neighborhood of Γ. Then $\mathcal{G}(\Gamma) \geq \mathcal{G}(\gamma)$. If $n = 3$, then it suffices to assume that K is constant on Γ.*

Letting γ in the above theorem be a geodesic sphere yields (1). For $n = 2$, the above theorem follows quickly from the Gauss-Bonnet theorem, without any assumptions on K. For $n \geq 3$, however, it is essential that K be constant on Γ due to examples by Dekster [8]. When Γ is smooth, and the constant in Theorem 1.1 is the supremum of K on the domain Ω bounded by Γ, then K is constant on

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Ω [14, Thm. 1.2], which reduces the above result to Borbély's theorem; see also [16, 21] for similar rigidity results, which extend "gap theorems" of Greene-Wu [18] and Gromov [4, Sec. 3]. Since the constant in Theorem 1.1 is arbitrary, not to mention that no regularity is assumed on Γ, we need to develop another approach.

We prove Theorem 1.1 via a comparison formula for total curvature of nested hypersurfaces [15]. This formula expresses the difference $\mathcal{G}(\Gamma) - \mathcal{G}(\gamma)$ as an integral over the region between the hypersurfaces, involving components of the Riemann curvature tensor R of M and derivatives of a function u whose level sets interpolate between γ and Γ. The key step is the choice of u, which is built from the distance functions of γ and Γ. Convexity of u ensures that the principal curvatures of its level sets are nonnegative, which control the sign of the leading terms in the comparison formula. When K is constant near Γ, the mixed terms of R vanish, yielding the desired monotonicity. In dimension three, a more delicate estimate shows that these mixed terms can still be controlled if K is constant only along Γ.

2. Preliminaries

Here we gather four lemmas which we need to prove Theorem 1.1. Throughout this work M is a Cartan-Hadamard n-manifold with sectional curvature K and Riemann curvature tensor R.

2.1. **The comparison formula.** Let Γ be a closed $\mathcal{C}^{1,1}$ hypersurface embedded in M. The Gauss-Kronecker curvature GK of Γ is the determinant of the second fundamental form of Γ with respect to the outward normal. The *total curvature* of Γ is given by

$$\mathcal{G}(\Gamma) := \int_\Gamma GK,$$

which is well-defined by Rademacher's theorem. A *domain* $\Omega \subset M$ is an open set with compact closure $\overline{\Omega}$. Let Ω be the domain bounded by Γ, and γ be another closed embedded $\mathcal{C}^{1,1}$ hypersurface which bounds a domain D with $\overline{D} \subset \Omega$. Then we say that γ is *nested* inside Γ. Suppose there exists a $\mathcal{C}^{1,1}$ function u on $\overline{\Omega} \setminus D$ with $\nabla u \neq 0$ on $\Omega \setminus \overline{D}$, which is constant on γ and Γ. We assume that $u|_\gamma < u|_\Gamma$ so that $e_n := \nabla u / |\nabla u|$ points outward along the level sets of u. Let κ_i be principal curvatures of the level sets with respect to e_n, and let $e_1, \dots, e_{n-1}$ form an orthonormal set of the corresponding principal directions. The *comparison formula*, first proved in [15] and developed further in [12, 17], states that

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) = -\int_{\Omega \setminus D} \sum_{1 \leq i \leq n-1} \widehat{GK}_i R_{inin} + \int_{\Omega \setminus D} \sum_{1 \leq i \neq j \leq n-1} \widehat{GK}_{ij} \frac{|\nabla u|_j}{|\nabla u|} R_{ijin},$$

where $|\nabla u|_j := \nabla_{e_j} |\nabla u|$, $R_{ijk\ell} = \langle R(e_i, e_j)e_k, e_\ell \rangle$ are components of the Riemann curvature of M, $\widehat{GK}_i$ denotes the product of all principal curvatures other than κ_i, and $\widehat{GK}_{ij}$ is the product without κ_i and κ_j. Note that $R_{inin} \leq 0$ because these are sectional curvatures of M. Furthermore, if u is a *convex function*, i.e., its composition with geodesics in M is convex, then $\kappa_i \geq 0$. Thus the first integral in the comparison formula is nonnegative, which immediately yields:

Lemma 2.1. *Let Γ, γ be $\mathcal{C}^{1,1}$ convex hypersurfaces in M, with γ nested inside Γ, and bounding domains D, Ω respectively. Then*

$$\mathcal{G}(\Gamma) - \mathcal{G}(\gamma) \geq \int_{\Omega \setminus D} \sum_{1 \leq i \neq j \leq n-1} \widehat{GK}_{ij} \frac{|\nabla u|_j}{|\nabla u|} R_{ijin}$$

for any $\mathcal{C}^{1,1}$ convex function u on $\overline{\Omega} \setminus D$ with $|\nabla u| \neq 0$, which is constant on γ and Γ with $u|_\gamma < u|_\Gamma$.

2.2. **The distance function.** For any set $X \subset M$, the *distance function* $d_X \colon M \to \mathbf{R}$ is defined by

$$d_X(p) := \inf_{x \in X} \mathrm{dist}_M(p, x),$$

where dist_M is the Riemannian distance in M. For basics of distance functions see [15, Sec. 2, 3] and references therein. In particular when X is convex, d_X is convex on M. Furthermore, d_X is locally $\mathcal{C}^{1,1}$ on $M \setminus \overline{X}$ [15, Prop. 2.7] and $|\nabla d_X| = 1$. A function $f \colon M \to \mathbf{R}$ is locally $\mathcal{C}^{1,1}$ on $X \subset M$ if it is $\mathcal{C}^{1,1}$ in local charts covering X. When f is $\mathcal{C}^1$, an equivalent condition is that ∇f be Lipschitz, i.e.,

$$\left|\nabla f(p) - \mathcal{T}_{q \to p} \nabla f(q)\right| \leq C \mathrm{dist}_M(p, q),$$

for all p, $q \in X$, where $\mathcal{T}_{q \to p}$ is parallel translation along the geodesic connecting q to p [3]. Throughout this work C denotes a positive constant whose value may change from one occurrence to the next. If f is locally $\mathcal{C}^{1,1}$ on a compact set X, then we say that f is $\mathcal{C}^{1,1}$ on X. The following fact is known in $\mathbf{R}^n$, see [10, Thm. 4.8 (5)&(9)] or [9, Thm. 6.3]. A set $X \subset M$ is *convex* if it contains the geodesic connecting any pair of its points.

Lemma 2.2. *Let X be a convex set in M. Then d_X^2 is locally $\mathcal{C}^{1,1}$ on M.*

Proof. Since $d_X = d_{\overline{X}}$, we may assume that X is closed. Also note that, since d_X is convex on M, and is $\mathcal{C}^1$ on $M \setminus \partial X$, the same holds for d_X^2. Let $\log_p \colon M \to T_pM$ be the inverse of the exponential map, $\pi_X \colon M \to X$ be the nearest point projection, and set $\overline{p} := \pi_X(p)$. Then

$$\nabla d_X(p) = -\frac{\log_p(\overline{p})}{d_X(p)}$$

for $p \in M \setminus X$ [15, Lem. 2.2]. Let $p_0 \in \partial X$, and set $f(p) := \mathrm{dist}_M^2(p, p_0)$. Since $0 \leq d_X^2(p) \leq f(p)$, and $f(p_0) = 0$, it follows that d_X^2 is differentiable at p_0 with $|\nabla d_X^2(p_0)| = 0$. Thus ∇d_X^2 is continuous, and so d_X^2 is $\mathcal{C}^1$ on M. Since $\nabla d_X^2(p) = 2 d_X(p) \nabla d_X(p) = -2 \log_p(\overline{p})$, for $p \in M \setminus X$, and $|\nabla d_X^2| = 0$ on X,

$$\nabla d_X^2(p) = -2 \log_p(\overline{p}),$$

for all $p \in M$. By the triangle inequality,

$$\begin{aligned} \left|\nabla d_X^2(p) - \mathcal{T}_{q \to p} \nabla d_X^2(q)\right| &= 2 \left|\log_p(\overline{p}) - \mathcal{T}_{q \to p} \log_q(\overline{q})\right| \\ &\leq 2 \left|\log_p(\overline{p}) - \log_p(\overline{q})\right| + 2 \left|\log_p(\overline{q}) - \mathcal{T}_{q \to p} \log_q(\overline{q})\right|. \end{aligned}$$

Since $K_M \leq 0$, $\log_p$ is nonexpansive. Furthermore, π_X is nonexpansive as well [6, Cor. 2.5]. Thus

$$\left|\log_p(\overline{p}) - \log_p(\overline{q})\right| \leq \mathrm{dist}_M(\overline{p}, \overline{q}) \leq \mathrm{dist}_M(p, q).$$

It now suffices to show that $\left|\log_p(\overline{q}) - \mathcal{T}_{q\to p}\log_q(\overline{q})\right| \leq C\mathrm{dist}_M(p,q)$, for p, q in any given compact set $Y \subset M$. More generally, for any fixed point o of M, and p, q in Y we claim that

$$\left|\log_p(o) - \mathcal{T}_{q\to p}\log_q(o)\right| \leq C\mathrm{dist}_M(p,q),$$

that is, the vector field $p \mapsto \log_p(o)$ is Lipschitz on Y. This is indeed the case because

$$\log_p(o) = -\frac{1}{2}\nabla \mathrm{dist}^2_M(p, o),$$

and $\mathrm{dist}^2_M(p, o)$ is smooth, which completes the proof. □

2.3. **Mixed curvature terms.** The Riemann curvature tensor R may be viewed as a symmetric bilinear form $\mathcal{R}$ on the space of 2-forms $\Lambda^2 TM$. More explicitly, let e_i be an orthonormal basis for T_pM. Then $e_i \wedge e_j$, for $1 \leq i < j \leq n$, form a basis for $\Lambda^2 T_pM$. There is a natural inner product on $\Lambda^2 T_pM$ given by $\langle e_i \wedge e_j, e_k \wedge e_\ell\rangle := \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$. In particular, $e_i \wedge e_j$ are orthonormal. We may then define $\mathcal{R}\colon \Lambda^2 T_pM \to \Lambda^2 T_pM$ by

$$\left\langle \mathcal{R}(e_i \wedge e_j), e_k \wedge e_\ell \right\rangle := R_{ijk\ell} = \left\langle R(e_i, e_j)e_k, e_\ell \right\rangle.$$

The *mixed curvature terms* are the coefficients $R_{ijk\ell}$ when $\{i, j\} \neq \{k, \ell\}$, or the off-diagonal components of $\mathcal{R}$. We say K is constant on $X \subset M$, or $K = k$ on X, if for all $p \in X$ and planes $\Pi \subset T_pM$, $K(\Pi) = k$.

Lemma 2.3. *Let $X \subset M$ be a compact set. Suppose that K is constant on X. Then there exists a neighborhood U of X such that for any orthonormal frame field on U, the absolute values of the mixed curvature terms on U are bounded above by Cd_X.*

Proof. Suppose that $K = k$ on X, and let e_i be a smooth orthonormal frame field on a neighborhood V of X. Let $\widetilde{\mathcal{R}}$ be the matrix representation of $\mathcal{R}$ with respect to $e_i \wedge e_j$. Then $\widetilde{\mathcal{R}} = kI$ on X, where I is the identity matrix. Since X is compact and $\widetilde{\mathcal{R}}$ is smooth, it follows that

$$|\widetilde{\mathcal{R}} - kI|_\infty \leq Cd_X,$$

on a neighborhood $U \subset V$, where $|\cdot|_\infty$ is the supremum of the absolute values of the coefficients. More explicitly, the above inequality follows from applying the mean value theorem to $\widetilde{\mathcal{R}} - kI$ restricted to geodesic segments originating from points of X. If $\widetilde{\mathcal{R}}'$ is the matrix representation of $\mathcal{R}$ with respect to $e'_i \wedge e'_j$, for any other frame field e'_i, then $\widetilde{\mathcal{R}}' = O^T\widetilde{\mathcal{R}}O$ for an orthogonal matrix O at each point. Thus

$$\left|\widetilde{\mathcal{R}}' - kI\right|_\infty = \left|O^T(\widetilde{\mathcal{R}} - kI)O\right|_\infty \leq C\left|\widetilde{\mathcal{R}} - kI\right|_\infty,$$

where C depends only on n. So $|\widetilde{\mathcal{R}}' - kI|_\infty \leq Cd_X$, which completes the proof. □

2.4. **Continuity of total curvature.** For a general convex hypersurface $\Gamma \subset M$, the total curvature $\mathcal{G}(\Gamma)$ is defined as follows. Let Ω be the domain bounded by Γ. The *outer parallel hypersurface* of Γ at distance $t \geq 0$ is given by $\Gamma_t := d_\Omega^{-1}(t)$. For $t > 0$, Γ_t is $\mathcal{C}^{1,1}$ [15, Lem. 2.6] and thus $\mathcal{G}(\Gamma_t)$ is well defined. We set

$$\mathcal{G}(\Gamma) := \lim_{t \searrow 0} \mathcal{G}(\Gamma_t).$$

By the comparison formula, $t \mapsto \mathcal{G}(\Gamma_t)$ is nondecreasing [17, Cor. 4.4]. Furthermore, since Γ_t is convex, $\mathcal{G}(\Gamma_t) \geq 0$. Thus $\mathcal{G}(\Gamma)$ is well-defined and finite. We record the following known fact [15, Note 3.7], which can be established via the theory of smooth valuations [1], and convergence of normal cycles [11, 23]. See [13] for a more direct proof.

Lemma 2.4 ([13])**.** *The total curvature functional $\mathcal{G}$ is continuous with respect to Hausdorff distance on the space of convex hypersurfaces $\Gamma \subset M$.*

3. Proof of Theorem 1.1

3.1. **The general case.** Let Ω, D be the domains bounded by Γ, γ, and d_Ω, d_D be the corresponding distance functions respectively. For $\lambda \in [0, \lambda_0]$, set

$$u^\lambda := \lambda d_D + d_\Omega^2.$$

Recall that d_D is locally $\mathcal{C}^{1,1}$ on $M \setminus \overline{D}$. Furthermore, d_Ω^2 is locally $\mathcal{C}^{1,1}$ on M by Lemma 2.2. Thus u^λ is locally $\mathcal{C}^{1,1}$ on $M \setminus \overline{D}$. Since d_D and d_Ω are convex, so is u^λ. Fix $\varepsilon > 0$ so small that the outer parallel hypersurface γ_ε is nested inside Γ. Let D_ε and Ω_ε be the domains bounded by γ_ε and Γ_ε respectively. Set

$$\Gamma_\varepsilon^\lambda := (u^\lambda)^{-1}(\varepsilon^2).$$

So $\Gamma_\varepsilon^\lambda \to \Gamma_\varepsilon$ as $\lambda \to 0$. In particular, choosing λ_0 sufficiently small, we may assume that $\Gamma_\varepsilon^\lambda$ lies in the annular region $\Omega_{2\varepsilon} \setminus \Omega$. Hence if $\Omega_\varepsilon^\lambda$ is the domain bounded by $\Gamma_\varepsilon^\lambda$, then $\Omega \subset \Omega_\varepsilon^\lambda \subset \Omega_{2\varepsilon}$. We may choose ε so small that K is constant on $\Omega_{2\varepsilon} \setminus \Omega$. Then the mixed curvature terms $R_{ijin} = 0$ on $\Omega_{2\varepsilon} \setminus \Omega$. Hence $\mathcal{G}(\Gamma_\varepsilon^\lambda) - \mathcal{G}(\gamma_\varepsilon) \geq 0$ by Lemma 2.1. Letting $\lambda \to 0$ followed by $\varepsilon \to 0$ completes the proof by Lemma 2.4.

3.2. **The case of $n = 3$.** When $n = 3$, by Lemma 2.1 we have

$$\mathcal{G}(\Gamma_\varepsilon^\lambda) - \mathcal{G}(\gamma_\varepsilon) \geq \int_{\Omega_\varepsilon^\lambda \setminus D_\varepsilon} F_\lambda, \qquad \text{where} \qquad F_\lambda := \sum_{1 \leq i,j \leq 2} \frac{|\nabla u^\lambda|_j}{|\nabla u^\lambda|} R_{ijin}.$$

Since $\Omega_\varepsilon^\lambda \setminus D_\varepsilon = (\Omega_\varepsilon^\lambda \setminus \Omega) \cup (\Omega \setminus D_\varepsilon)$, $u^\lambda = \lambda d_D$ on $\Omega \setminus D_\varepsilon$, and $|\nabla d_D| = 1$, it follows that F_λ vanishes identically on $\Omega \setminus D_\varepsilon$. Thus

$$\mathcal{G}(\Gamma_\varepsilon^\lambda) - \mathcal{G}(\gamma_\varepsilon) \geq \int_{\Omega_\varepsilon^\lambda \setminus \Omega} F_\lambda \geq -\left| \int_{\Omega_\varepsilon^\lambda \setminus \Omega} F_\lambda \right| \geq -\int_{\Omega_\varepsilon^\lambda \setminus \Omega} |F_\lambda| \geq -\int_{\Omega_{2\varepsilon} \setminus \Omega} |F_\lambda|.$$

Next we show that $|F_\lambda|$ is uniformly bounded above (almost everywhere) on $\Omega_{2\varepsilon} \setminus \Omega$, by the following three estimates. Since $\nabla u^\lambda = \lambda \nabla d_D + \nabla d_\Omega^2$ is uniformly Lipschitz,

$$\big| |\nabla u^\lambda|_j \big| \leq C,$$

for C independent of λ. By Lemma 2.3, we also have

$$|R_{ijin}| \leq C d_\Omega,$$

where again C does not depend on λ. Next note that $\langle \nabla d_\Omega, \nabla d_D \rangle \geq 0$, because level sets of d_Ω are convex, and ∇d_D is tangent to geodesic rays which originate in Ω. Thus

$$\left|\nabla u^\lambda\right| = \sqrt{4d_\Omega^2 + \lambda^2 + 4\lambda d_\Omega \langle \nabla d_\Omega, \nabla d_D \rangle} \geq 2d_\Omega.$$

So we conclude that $|F_\lambda| \leq C$ on $\Omega_{2\varepsilon} \setminus \Omega$, which yields

$$\mathcal{G}(\Gamma_\varepsilon^\lambda) - \mathcal{G}(\gamma_\varepsilon) \geq -C|\Omega_{2\varepsilon} \setminus \Omega|.$$

Again letting $\lambda \to 0$ followed by $\varepsilon \to 0$ completes the proof by Lemma 2.4.

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