SHADES AND CONVEXITY OF HYPERSURFACES

MOHAMMAD GHOMI

Abstract. Let $f\colon M^n \to \mathbf{R}^{n+1}$, $n \geq 2$, be a smooth immersion of a closed oriented manifold, with Gauss map $\nu\colon M \to \mathbf{S}^n$. For each unit vector u, the *shade* $S_u \subset M$ is the set of points where ν makes an acute angle with u. We show that f is a convex embedding if and only if every shade is contractible. More generally, if every shade is k-connected, then f is a convex embedding provided that $n \leq 2k+1$, or $n = 2k+2$ and $H_{k+1}(M;\mathbf{Z}) = 0$. On the other hand, for every $n \geq 2k+3$ there is a nonconvex embedding $f\colon \mathbf{S}^n \to \mathbf{R}^{n+1}$ all of whose shades are k-connected. These results give sharp answers to a question of H. Wente in all dimensions.

1. Introduction

Let M^n, $n \geq 2$, be a closed oriented manifold, and $f\colon M \to \mathbf{R}^{n+1}$ be a smooth immersion with Gauss map $\nu\colon M \to \mathbf{S}^n$. For $u \in \mathbf{S}^n$, the *shade* [8] (a.k.a. shadow [7]) of M in direction u is given by

$$S_u := \big\{p \in M : \langle \nu(p), u\rangle > 0\big\}.$$

If f is a *convex embedding*, i.e., $f(M)$ bounds a convex set, then S_u is the region of M which lies in the dark when M is illuminated by light rays parallel to u. In particular S_u is connected. Motivated by stability problems for surfaces of constant mean curvature [4, 7], H. Wente asked in 1978 whether connectedness of the shades forces f to be a convex embedding. The author showed [6] that for $n = 2$ the answer is yes if the shades are simply connected, or else $M \cong \mathbf{S}^2$, i.e., M is homeomorphic to $\mathbf{S}^2$; otherwise there are counterexamples. Here we extend those results to higher dimensions. A space is *k-connected* if it is path connected and its homotopy groups $\pi_i = 0$ for $1 \leq i \leq k$.

Theorem 1.1. *The immersion f is a convex embedding if and only if every shade is contractible. More generally, if every shade is k-connected, then f is a convex embedding, provided that $n \leq 2k+1$, or $n = 2k+2$ and $H_{k+1}(M;\mathbf{Z}) = 0$.*

When $k \leq 1$ and every shade is k-connected, the hypothesis $H_{k+1}(M;\mathbf{Z}) = 0$ for $n = 2k+2$ is equivalent to $M \cong \mathbf{S}^n$; see Note 2.9. So if $n \leq 3$ or $M \cong \mathbf{S}^4$, it suffices that every shade be simply connected; and if $M \cong \mathbf{S}^2$, it suffices that every shade be connected. The dimension restrictions in Theorem 1.1 are sharp:

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Theorem 1.2. *For every $k \geq 0$ and $n \geq 2k+3$, there is a smooth nonconvex embedding $f\colon \mathbf{S}^n \to \mathbf{R}^{n+1}$ such that every shade is k-connected.*

Thus, when $M \cong \mathbf{S}^n$, k-connected shades force f to be a convex embedding precisely when $n \leq 2k+2$. The proof of Theorem 1.1 is based on the equality case of the Chern–Lashof theorem. A lifting principle for projections of open sets translates contractibility of the shades into a count of the preimages of regular values of the Gauss map, which yields convexity. When the shades are only assumed to be k-connected, we first show that they are contractible, via Alexander duality. The examples of Theorem 1.2 are certain embeddings of the join $\mathbf{S}^p * \mathbf{S}^q$ in $\mathbf{R}^{p+1} \oplus \mathbf{R}^{q+1}$ which are invariant under rotations of both factors.

2. Convexity Characterizations

Throughout we assume that M is a *closed* manifold, i.e., compact, connected without boundary, of dimension $n \geq 2$. In this section we establish Theorem 1.1. The proof is based on the Chern–Lashof theorem [2, 3, 12, 13], which states that the *total absolute curvature*

$$\tau(f) \coloneqq \int_M |\det d\nu|$$

satisfies $\tau(f) \geq \operatorname{vol}(\mathbf{S}^n)$, with equality if and only if f is a convex embedding. By Sard's theorem and the area formula, $\tau(f) = \int_{\mathbf{S}^n} \#\nu^{-1}(u)$, which is equal to $\operatorname{vol}(\mathbf{S}^n)$ when every regular value of ν has exactly one preimage. So we obtain the following criterion.

Lemma 2.1. *If every regular value of ν has exactly one preimage, then f is a convex embedding.*

2.1. **Contractible shades.** First we show that contractibility of the shades forces every regular value of ν to have exactly one preimage. To this end we write $S_u = \{\sigma_u > 0\}$, where $\sigma_u \coloneqq \langle \nu, u \rangle$ is the *shading function*. Let

$$E \coloneqq \big\{(p,u) \in M \times \mathbf{S}^n : \sigma_u(p) > 0\big\},$$

and $\pi\colon E \to \mathbf{S}^n$ be the projection into the second factor. Then $\pi^{-1}(u) = S_u \times \{u\}$, so the shades of M form the fibers of π. Furthermore, π is surjective, i.e., every shade is nonempty. Indeed σ_u is not identically zero, since $\nu = \pm u$ at a maximum point of the height function $\langle f, u \rangle$, and so $\sigma_u = \pm 1 \neq 0$ there. On the other hand $\int_M \sigma_u = 0$ by Stokes' theorem, since σ_u times the volume form of M is the pullback under f of the n-form $\Omega(u, \cdot, \ldots, \cdot)$ on $\mathbf{R}^{n+1}$, where Ω is the volume form; this n-form is closed, and hence exact. So σ_u changes sign. Although π need not be a fiber bundle, the openness of E still admits enough local lifting to relate the topology of S_u to that of E:

Lemma 2.2. *Let X and B be manifolds, $E \subset X \times B$ be open, and $\pi\colon E \to B$ be the second projection. Assume π is surjective. If the fibers of π are k-connected, then π induces bijections on π_i for $0 \leq i \leq k$. If the fibers are contractible, then π is a homotopy equivalence.*

Proof. Let $\mathbf{B}^m$ denote the closed unit ball in $\mathbf{R}^m$. Given a lifting problem over $[0,1] \times \mathbf{B}^m$ with a lift at time zero, openness of E allows one to keep the X-coordinate fixed for

a short time. Compactness of $\mathbf{B}^m$ gives a uniform positive time. Thus π is a Serre microfibration. Recall that a map is *m-connected* if it induces bijections on π_i for $i < m$, and a surjection on π_m. If the fibers of π are k-connected, then each of them is mapped to a point by a $(k+1)$-connected map. So [15, Thm. 1.3], applied with q the identity map of B, shows that π is $(k+1)$-connected, which yields the first assertion. If the fibers are contractible, then π is a Serre fibration by a lemma of Weiss [17, Lem. 2.2], and so is a weak homotopy equivalence, which is a homotopy equivalence by Whitehead's theorem, since manifolds have the homotopy type of CW complexes [14]. □

For $u \in \mathbf{S}^n$, let $H_u \coloneqq \{x \in \mathbf{S}^n : \langle x, u\rangle > 0\}$ be the open hemisphere centered at u. Then $S_u = \nu^{-1}(H_u)$, and the fiber of E over each $p \in M$ is $H_{\nu(p)}$. We also put $I_u \coloneqq E \cap (M \times H_u)$.

Proposition 2.3. *If every shade is contractible, then every regular value of ν has exactly one preimage.*

Proof. Let $-u$ be a regular value of ν, so that $\nu^{-1}(-u)$ is finite, and put $X \coloneqq M \setminus \nu^{-1}(-u)$. Projection of I_u to H_u has contractible fibers S_v. So Lemma 2.2 makes it a homotopy equivalence, and since H_u is contractible, so is I_u. Projection of I_u to M, on the other hand, has fiber $H_u \cap H_{\nu(p)}$ over each $p \in M$, which is geodesically convex, and is nonempty exactly when $\nu(p) \neq -u$. So the image of this projection is X, and its fibers are contractible. Thus Lemma 2.2 shows that X is contractible as well. Write $N \coloneqq \#\nu^{-1}(-u)$. Since X is contractible, the long exact sequence of (M, X) yields $H_n(M, X; \mathbf{Z}) \cong H_n(M; \mathbf{Z}) \cong \mathbf{Z}$. On the other hand, $H_n(M, X; \mathbf{Z}) \cong \mathbf{Z}^N$ by excision. Thus $N = 1$. □

Corollary 2.4. *f is a convex embedding if and only if every shade is contractible.*

Proof. If every shade is contractible, then f is a convex embedding by Proposition 2.3 and Lemma 2.1. Conversely, each shade of a convex embedding is a graph over an open convex set, and is therefore contractible; see [6]. □

2.2. **k-connected shades.** Now we show that k-connected shades are contractible under the hypotheses of Theorem 1.1; so Corollary 2.4 applies. We need three preliminary lemmas. For $u \in \mathbf{S}^n$ and $0 < c < 1$, we define the *enlarged shades*

$$S_{u,c} \coloneqq \{\sigma_u > -c\}.$$

Lemma 2.5. *If every shade is k-connected, then so is every enlarged shade.*

Proof. Let $D \subset \mathbf{S}^n$ be the open cap of angular radius $\arcsin c$ centered at u, and put $Y \coloneqq E \cap (M \times D)$. Projection of Y to D has fiber S_v over each $v \in D$. Since D is contractible, Lemma 2.2 shows that Y is k-connected. Projection of Y to M, on the other hand, has fiber $D \cap H_{\nu(p)}$ over each $p \in M$, which is geodesically convex, and is nonempty exactly when the angle between $\nu(p)$ and u is less than $\pi/2 + \arcsin c$, i.e., when $\sigma_u(p) > -c$. So the image of this projection is $S_{u,c}$, and its fibers are contractible. Thus Lemma 2.2 shows that $S_{u,c}$ is k-connected as well. □

Lemma 2.6. *If every shade is k-connected, then M is $\min\{k, n-1\}$-connected. Furthermore, if $n \leq 2k+2$, then $H_i(M; \mathbf{Z}) = 0$ for $1 \leq i \leq n-1$, except possibly for $i = k+1$ when $n = 2k+2$.*

Proof. Let $r \coloneqq \min\{k, n-1\}$. Projection of E to M has contractible fibers $H_{\nu(p)}$, and projection of E to $\mathbf{S}^n$ has k-connected fibers. So Lemma 2.2 yields $\pi_i(M) \cong \pi_i(E) \cong \pi_i(\mathbf{S}^n) = 0$ for $1 \leq i \leq r$, which proves the first assertion. Thus $H_i(M; \mathbf{Z}) = 0$ for $1 \leq i \leq r$ by the Hurewicz theorem, and then $H^i(M; \mathbf{Z}) = 0$ for $1 \leq i \leq r$ by the universal coefficient theorem. So Poincaré duality yields $H_i(M; \mathbf{Z}) = 0$ for $n - r \leq i \leq n-1$. Now suppose that $n \leq 2k+2$. If $k \geq n-1$, we are done. Otherwise $r = k$, and the only degrees not covered are $k+1 \leq i \leq n-k-1$. This range is empty if $n \leq 2k+1$, and consists of $i = k+1$ if $n = 2k+2$. □

Lemma 2.7. *Let $S \subset M$ be a nonempty proper open subset, and $S' \coloneqq M \setminus S$. Suppose that S is k-connected, S' has arbitrarily small k-connected open neighborhoods, and $H_i(M; \mathbf{Z}) = 0$ for $1 \leq i \leq n-1$. If $n \leq 2k+2$, then S is contractible.*

Proof. By Alexander duality [16, Thm. 6.2.17], $H_j(M, S; \mathbf{Z})$ is isomorphic to the direct limit of $H^{n-j}(U; \mathbf{Z})$ over open neighborhoods U of S'. Since $H_i(M; \mathbf{Z}) = 0$ for $1 \leq i \leq n-1$, the long exact sequence of (M, S) then shows that $\tilde{H}_i(S; \mathbf{Z})$ is isomorphic to the direct limit of $\tilde{H}^{n-i-1}(U; \mathbf{Z})$, for $0 \leq i \leq n-1$. By hypothesis there is a cofinal family of such U which are k-connected, so that $\tilde{H}^m(U; \mathbf{Z}) = 0$ for $m \leq k$. Thus $\tilde{H}_i(S; \mathbf{Z}) = 0$ for $n-k-1 \leq i \leq n-1$, while $\tilde{H}_i(S; \mathbf{Z}) = 0$ for $i \leq k$ since S is k-connected, and $H_i(S; \mathbf{Z}) = 0$ for $i \geq n$ since S is a noncompact n-manifold. As $n \leq 2k+2$, these ranges cover all degrees, so S is acyclic. Furthermore S is simply connected if $k \geq 1$, and also if $k = 0$, since then $n = 2$, the fundamental group of a noncompact surface is free, and $H_1(S; \mathbf{Z}) = 0$ is its abelianization. Hence S is contractible by the Hurewicz and Whitehead theorems [11], since it has the homotopy type of a CW complex [14]. □

Proposition 2.8. *Suppose that every shade is k-connected, and either $n \leq 2k+1$, or $n = 2k+2$ and $H_{k+1}(M; \mathbf{Z}) = 0$. Then every shade is contractible.*

Proof. By Lemma 2.6, $H_i(M; \mathbf{Z}) = 0$ for $1 \leq i \leq n-1$. Now fix $u \in \mathbf{S}^n$. Then $S'_u = \{\sigma_u \leq 0\}$ is nonempty because $S_{-u} \neq \varnothing$, and its neighborhoods $\{\sigma_u < c\} = S_{-u,c}$, $0 < c < 1$, shrink to S'_u and are k-connected by Lemma 2.5. So Lemma 2.7 shows that S_u is contractible. □

2.3. **Proof of Theorem 1.1.** The first assertion is Corollary 2.4, and the second follows from Corollary 2.4 together with Proposition 2.8. □

Note 2.9. When $k = 0$, the hypothesis $H_{k+1}(M; \mathbf{Z}) = 0$ in Theorem 1.1 cannot be dropped, by the counterexamples in [6, 10], which are constructed using "skew loops" [9]. For $k \geq 1$, whether it can be dropped when $n = 2k+2$ remains open. In the latter case Lemma 2.6 shows that M is k-connected, and $H_{k+1}(M; \mathbf{Z})$ is the only homology group of M in degrees 1 through $n-1$ which may not vanish. So $H_{k+1}(M; \mathbf{Z}) \cong H^{k+1}(M; \mathbf{Z}) \cong \mathrm{Hom}(H_{k+1}(M; \mathbf{Z}), \mathbf{Z})$ by Poincaré duality and the universal coefficient theorem; thus $H_{k+1}(M; \mathbf{Z})$ is free, and the hypothesis is equivalent to the vanishing of the Betti number $b_{k+1}(M)$. When $k = 0$, it is also equivalent to $M \cong \mathbf{S}^2$, by the

classification of surfaces. When $k = 1$, it is equivalent to $M \cong \mathbf{S}^4$, since M is then a simply connected homology 4-sphere, hence a homotopy sphere, and so is homeomorphic to $\mathbf{S}^4$ by Freedman's theorem [5]. The simplest candidate for a counterexample is $M = \mathbf{S}^2 \times \mathbf{S}^2$, which is the union of two copies of $\mathbf{S}^2 \times \mathbf{B}^2$ along $\mathbf{S}^2 \times \mathbf{S}^1$; we do not know whether it admits an embedding in $\mathbf{R}^5$ with simply connected shades.

3. Nonconvex Embeddings

In this section we prove Theorem 1.2. Our examples will be parametrized by spherical joins [1, 11]. For $p, q \geq 1$, the *spherical join* of $\mathbf{S}^p$ and $\mathbf{S}^q$ is

$$\mathbf{S}^p * \mathbf{S}^q := \big([0, \pi/2] \times \mathbf{S}^p \times \mathbf{S}^q\big)/\sim,$$

where points of the product are denoted by (t, x, y), and $(0, x, y) \sim (0, x, y')$, $(\pi/2, x, y) \sim (\pi/2, x', y)$; so the $\mathbf{S}^q$ factor is collapsed at $t = 0$, and the $\mathbf{S}^p$ factor at $t = \pi/2$. The equivalence class of (t, x, y) is denoted by $[t, x, y]$. The standard map $[t, x, y] \mapsto (\cos t\, x, \sin t\, y)$ identifies $\mathbf{S}^p * \mathbf{S}^q$ smoothly with the unit sphere of $\mathbf{R}^{p+1} \oplus \mathbf{R}^{q+1}$. Theorem 1.2 follows from the next result by setting $p := k + 1$ and $q := n - k - 2$, which are at least $k + 1$ when $n \geq 2k + 3$.

Theorem 3.1. *Let $p, q \geq 1$. There is a smooth nonconvex embedding $f\colon \mathbf{S}^p * \mathbf{S}^q \to \mathbf{R}^{p+q+2}$ whose shades are all $(\min\{p, q\} - 1)$-connected.*

We construct f and establish its nonconvexity in Section 3.1. The topology of its shades is studied in Section 3.2, and the proof is completed in Section 3.3.

3.1. **The construction.** Fix $p, q \geq 1$, and put $n := p + q + 1$ and $M := \mathbf{S}^p * \mathbf{S}^q$. Near the endpoints of $[0, \pi/2]$ we use the standard charts

$$(x, z) \in \mathbf{S}^p \times \mathbf{R}^{q+1}, \quad z = ty, \qquad (w, y) \in \mathbf{R}^{p+1} \times \mathbf{S}^q, \quad w = (\pi/2 - t)x,$$

where $|z|, |w| < \pi/2$. In the above identification of $\mathbf{S}^p * \mathbf{S}^q$ with $\mathbf{S}^n$, the Gauss map is $[t, x, y] \mapsto (\cos t\, x, \sin t\, y)$. The idea is to replace t in this formula by a nonmonotone function $\theta(t)$, and then integrate to find a hypersurface with the resulting Gauss map. Let $\theta\colon \mathbf{R} \to \mathbf{R}$ be a smooth odd function such that $\theta(\pi/2 - t) = \pi/2 - \theta(t)$, $0 < \theta(t) < \pi/2$ for $0 < t < \pi/2$, and $\theta'(t_0) < 0$ for some $t_0 \in (0, \pi/2)$. For instance we may set $\theta(t) := t + \frac{1}{3}\sin(4t)$. Also let $a, b\colon [0, \pi/2] \to \mathbf{R}$ be the functions determined by

$$a'(t) = -\sin\theta(t), \quad a(\pi/2) = 0, \qquad b'(t) = \cos\theta(t), \quad b(0) = 0.$$

Then $a' < 0$ and $b' > 0$ on $(0, \pi/2)$; in particular a and b are positive on $(0, \pi/2)$. Define

$$f([t, x, y]) := (a(t)x,\, b(t)y) \in \mathbf{R}^{p+1} \oplus \mathbf{R}^{q+1}.$$

Lemma 3.2. *f is a smooth embedding with the Gauss map*

$$\nu([t, x, y]) := (\cos\theta(t)\, x, \sin\theta(t)\, y).$$

Proof. For $0 < t < \pi/2$, $f_t = (-\sin\theta(t)\, x, \cos\theta(t)\, y)$, while $f_\xi = (a(t)\xi, 0)$ and $f_\eta = (0, b(t)\eta)$ for $\xi \in T_x\mathbf{S}^p$ and $\eta \in T_y\mathbf{S}^q$. These vectors are linearly independent because $a(t), b(t) > 0$ and $|f_t| = 1$. The vector $\nu([t, x, y])$ has unit length and is orthogonal to

all of them. Endpoint smoothness follows from parity. Since θ is odd, there are smooth one-variable functions $\alpha, \beta, \kappa, \delta$ such that

$$a(0) - a(t) = t^2\alpha(t^2), \qquad b(t) = t\beta(t^2), \qquad \cos\theta(t) = \kappa(t^2), \qquad \sin\theta(t) = t\delta(t^2),$$

with $\beta(0) = 1$. In the chart $z = ty$,

$$f(x,z) = \big(\big(a(0) - |z|^2\,\alpha(|z|^2)\big)x,\ \beta(|z|^2)z\big), \qquad \nu(x,z) = \big(\kappa(|z|^2)x,\ \delta(|z|^2)z\big).$$

Thus both maps extend smoothly across $z = 0$, and $df_{(x,0)}(\xi, v) = (a(0)\xi, v)$ has full rank. At the other endpoint write $s = \pi/2 - t$. The symmetry of θ gives $a(\pi/2 - s) = b(s)$ and $b(\pi/2 - s) = a(s)$. The same parity calculation in the chart $w = sx$ gives smooth extensions and $df_{(0,y)}(v, \eta) = (v, b(\pi/2)\eta)$, again of full rank. Finally, the norms of the two Euclidean blocks of $f([t, x, y])$ are $a(t)$ and $b(t)$. Their strict monotonicity determines t, after which every noncollapsed spherical variable is determined. Thus f is injective, and so is an embedding since M is compact. □

Lemma 3.3. *f is not convex.*

Proof. Let $\mathrm{II}(V, W) \coloneqq \langle d\nu(V), df(W)\rangle$ be the second fundamental form of f. On $0 < t < \pi/2$, $d\nu(\partial_t) = \theta'(t) f_t$, so $\mathrm{II}(\partial_t, \partial_t) = \theta'(t)$. For $\xi \in T_x\mathbf{S}^p$ and $\eta \in T_y\mathbf{S}^q$,

$$\mathrm{II}(\xi, \xi) = a(t)\cos\theta(t)\,|\xi|^2, \qquad \mathrm{II}(\eta, \eta) = b(t)\sin\theta(t)\,|\eta|^2.$$

So the principal curvatures of f are $\theta'(t)$, $\cos\theta(t)/a(t)$, $\sin\theta(t)/b(t)$. At $t = t_0$, the last two are positive while $\theta'(t_0) < 0$. So II is indefinite. □

3.2. **Topology of the shades.** Fix $u = (u_1, u_2) \in \mathbf{R}^{p+1} \oplus \mathbf{R}^{q+1}$, so that $|u_1|^2 + |u_2|^2 = 1$, and choose $e_1 \in \mathbf{S}^p$ and $e_2 \in \mathbf{S}^q$ so that $u_i = |u_i|\, e_i$; if $u_i = 0$, choose e_i arbitrarily. Put $x_1 \coloneqq \langle x, e_1\rangle$ and $y_2 \coloneqq \langle y, e_2\rangle$. Then the shading function is

$$\sigma_u([t, x, y]) = |u_1|\cos\theta(t)\, x_1 + |u_2|\sin\theta(t)\, y_2. \tag{1}$$

Proposition 3.4. *The shade S_u is $(\min\{p, q\} - 1)$-connected.*

Proof. Let $0 \le j < \min\{p, q\}$, and $g\colon \mathbf{S}^j \to S_u$ be a continuous map. The spheres $\{t = 0\}$ and $\{t = \pi/2\}$ have codimensions $q+1$ and $p+1$ in M, and in their complement $\{0 < t < \pi/2\} \cong (0, \pi/2) \times \mathbf{S}^p \times \mathbf{S}^q$ the sets $\{x = -e_1\}$ and $\{y = -e_2\}$ have codimensions p and q. All these exceed j. So by transversality, we may assume, after a small homotopy in S_u, that g avoids these sets, i.e., $g = [t, x, y]$ with $0 < t < \pi/2$, $x \ne -e_1$, and $y \ne -e_2$. Then the shortest great circle arcs from x to e_1, and from y to e_2, depend continuously on the point of $\mathbf{S}^j$. Moving x and y along these arcs, keeping t fixed, does not decrease x_1 or y_2, and so by (1) does not decrease σ_u. Thus g is homotopic in S_u to a map into the central ridge $\{[t, e_1, e_2] : 0 < t < \pi/2\}$, where $\sigma_u = |u_1|\cos\theta(t) + |u_2|\sin\theta(t) > 0$. Finally, moving t linearly to $\pi/4$ contracts this map within the ridge. Hence $\pi_j(S_u) = 0$. □

3.3. **Proof of Theorem 3.1.** Let f be the map constructed in Section 3.1. Then f is a smooth embedding by Lemma 3.2, which is not convex by Lemma 3.3, and its shades are $(\min\{p, q\} - 1)$-connected by Proposition 3.4. □

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332
Email address: ghomi@math.gatech.edu
URL: ghomi.math.gatech.edu