TOTAL CURVATURE AND ISOPERIMETRIC INEQUALITIES IN PINCHED CARTAN–HADAMARD MANIFOLDS

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Abstract. We establish a sharp lower bound for the total Gauss–Kronecker curvature of convex hypersurfaces in Cartan–Hadamard manifolds with pinched negative curvature. The bound holds in all dimensions when the diameter is small relative to the curvature scale, and in dimensions 4 and 5 without any restriction on the diameter, provided that the pinching is sufficiently tight. The proofs are based on the Chern–Gauss–Bonnet theorem and weighted Hsiung–Minkowski inequalities. As an application, we obtain the isoperimetric inequality of the Cartan–Hadamard conjecture in dimension 5 under sufficiently pinched curvature.

1. Introduction

A Cartan–Hadamard manifold M^n is a complete simply connected Riemannian space with sectional curvature $K \leq 0$. A convex hypersurface $\Gamma \subset M$ is the boundary of a compact convex set with nonempty interior. If Γ is smooth, its total Gauss–Kronecker curvature $\mathcal{G}(\Gamma)$ is the integral of the determinant of its second fundamental form. In hyperbolic space $\mathbf{H}^n$,

$$\mathcal{G}(\Gamma) \geq |\mathbf{S}^{n-1}|, \tag{1}$$

where $|\mathbf{S}^{n-1}|$ denotes the volume of the unit sphere [8, 18]. For $n \leq 3$ the same inequality holds in every Cartan–Hadamard manifold, as an immediate consequence of the Gauss–Bonnet theorem and the Gauss equation. Whether (1) holds in higher dimensions is a long-standing problem [4, 43], which implies the Cartan–Hadamard conjecture [3, 9, 25] on the isoperimetric inequality [18]. We establish (1) when $\operatorname{diam}(\Gamma) \coloneqq \sup_{x,y\in\Gamma} \operatorname{dist}_M(x,y)$ is sufficiently small relative to the bounds of K:

Theorem 1.1. *For $n \geq 4$, let M^n be a Cartan–Hadamard manifold with curvature $-b^2 \leq K \leq -a^2 < 0$. Then there exists $\delta_n > 0$, depending only on n, such that (1) holds for every smooth convex hypersurface $\Gamma \subset M$ with $\operatorname{diam}(\Gamma) \leq \delta_n a/b^2$.*

The proof yields a stronger quantitative estimate; see Proposition 5.1. In dimensions 4 and 5 the diameter restriction can be removed as follows. Set

$$\lambda_4 \coloneqq \frac{1}{\sqrt{3}} \approx 0.58, \qquad \lambda_5 \coloneqq \frac{1+\sqrt{113}}{14} \approx 0.83.$$

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Theorem 1.2. *For $n = 4, 5$, let M^n be a Cartan–Hadamard manifold with curvature $-b^2 \leq K \leq -\lambda_n b^2$, for some $b > 0$. Then (1) holds for every smooth convex hypersurface $\Gamma \subset M^n$.*

Again the proof yields explicit lower bounds for the defect in (1); see Propositions 6.1 and 6.3. These theorems provide the first instances of (1) in dimensions greater than 3 for Cartan–Hadamard manifolds of nonconstant negative curvature near Γ. In light of the reduction of the isoperimetric inequality to the total curvature inequality [18, Thm. 7.1], Theorem 1.2 immediately yields the following result, which gives the first instance of the Cartan–Hadamard conjecture in a dimension greater than 4 with nonconstant negative curvature. For any set $S \subset M^n$, let $|S|$ and $|\partial S|$ denote its volume and perimeter respectively.

Corollary 1.3. *Let M^5 be a Cartan–Hadamard manifold with curvature $-b^2 \leq K \leq -\lambda_5 b^2$, for some $b > 0$. Then every bounded set $\Omega \subset M$ of finite perimeter and positive volume satisfies $|\partial\Omega| > |\partial B|$, where B is a ball in $\mathbf{R}^5$ with $|B| = |\Omega|$.*

For general Cartan–Hadamard manifolds, the isoperimetric inequality has been established only in dimensions $n \leq 4$ [12, 30, 42], while the total curvature inequality (1) is known only for $n \leq 3$. For domains of small volume, related isoperimetric inequalities are known under additional compactness, bounded-geometry, or asymptotic hypotheses [13, 35, 36]. The pinching assumptions above do not force K to be constant: for all $0 < a < b$ there exist Cartan–Hadamard manifolds with $-b^2 \leq K \leq -a^2$ which are not of constant curvature, such as the universal covers of the closed manifolds constructed by Gromov–Thurston [24]. The constant in (1) is sharp, since the total curvature of geodesic spheres converges to $|\mathbf{S}^{n-1}|$ as their radii shrink to zero [18, Lem. 5.1]. The total curvature inequality (1) has been established recently in the case where M^n has nullity at least $n-3$ [16] or the curvature is constant near Γ [22]. See [19, 21, 40] for some other results in this area, and [18, 31] for more references and background.

The proofs combine two main ingredients. First, we obtain an expression for the defect $\mathcal{G}(\Gamma) - |\mathbf{S}^{n-1}|$ by expanding the Chern–Gauss–Bonnet integrands according to the number of ambient curvature factors (Section 3). The term linear in the ambient curvature has a favorable sign, while the higher-order terms must be controlled by total mean curvatures. Second, we develop a weighted Hsiung–Minkowski identity in terms of Newton transformations and apply it, together with Hessian comparison and Stokes theorem, to suitable radial functions (Section 4). The resulting inequalities provide the required controls. Choosing the squared distance from an interior point of the convex body bounded by Γ yields the estimates used to absorb the higher-order terms when the diameter is small, and leads to Theorem 1.1 (Section 5). Choosing the distance from an exterior point gives diameter-free estimates, which lead to Theorem 1.2 (Section 6).

2. Algebraic Preliminaries

We begin by collecting the pointwise estimates needed for curvature tensors in the proofs below. An *algebraic curvature tensor* on $\mathbf{R}^n$ is a 4-tensor R satisfying the symmetries

$$R(x,y,z,w) = R(z,w,x,y), \qquad R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z),$$

$$R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0.$$

For orthonormal vectors x, y, the *sectional curvature* of R is defined by

$$K(x,y) \coloneqq R(x,y,x,y).$$

Furthermore, the *Ricci* and *scalar curvatures* are given by

$$\mathrm{Ric}_R(x,y) \coloneqq \sum_{i=1}^{n} R(e_i, x, e_i, y), \qquad \mathrm{Scal}_R \coloneqq \sum_{i=1}^{n} \mathrm{Ric}_R(e_i, e_i),$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbf{R}^n$. We write $R_{ijk\ell} \coloneqq R(e_i, e_j, e_k, e_\ell)$, and $K_{ij} \coloneqq K(e_i, e_j) = R_{ijij}$. The inner product and norm on tensors are those induced by the Euclidean inner product on $\mathbf{R}^n$. Thus, for k-tensors S and T, we have

$$\langle S, T \rangle \coloneqq \sum_{i_1, \ldots, i_k} S(e_{i_1}, \ldots, e_{i_k}) T(e_{i_1}, \ldots, e_{i_k}), \qquad |T| \coloneqq \langle T, T \rangle^{\frac{1}{2}}.$$

Throughout the paper, c_n denotes a positive constant depending only on n, which may change from line to line.

2.1. **General estimates.** First we record the curvature estimates which hold in all dimensions. Let $\mathcal{R}_n$ denote the space of algebraic curvature tensors on $\mathbf{R}^n$, and let R_k be the tensor of constant sectional curvature k.

Lemma 2.1. *Let $R \in \mathcal{R}_n$ with $-1 \leq K \leq -\lambda$. Then $|R - R_{-1}| \leq c_n(1-\lambda)$.*

Proof. Let $\|R\|_K \coloneqq \sup_{x,y} |K(x,y)|$, where the supremum is over all orthonormal pairs $x, y \in \mathbf{R}^n$. An algebraic curvature tensor is determined by its sectional curvatures through polarization [10, p. 14]. Thus $\|R\|_K = 0$ implies $R = 0$. Since $\|\cdot\|_K$ is also homogeneous and satisfies the triangle inequality, it is a norm on $\mathcal{R}_n$. As $\mathcal{R}_n$ is finite-dimensional, this norm is equivalent to the standard tensor norm. Since the sectional curvature of $R - R_{-1}$ lies between 0 and $1 - \lambda$, the result follows. $\square$

We also need the following estimate of Berger [5, 29].

Lemma 2.2 (Berger)**.** *Let $R \in \mathcal{R}_n$ with $0 \leq K \leq \Lambda$. Then $|R_{ijjk}| \leq \Lambda/2$ for distinct i, j, k.*

For even n, define the *Pfaffian* of $R \in \mathcal{R}_n$ by

$$\mathrm{Pf}(R) \coloneqq \frac{1}{2^{n/2} n!} \sum \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{n-1} i_n j_{n-1} j_n}, \tag{2}$$

where the sum is over all indices from 1 to n, and $\epsilon_{i_1 \cdots i_n}$ is the Levi-Civita symbol, equal to the sign of the permutation $(i_1, \ldots, i_n)$ when the indices are distinct, and to zero otherwise. The normalizing constant ensures that $\mathrm{Pf}(R_1) = 1$.

For a symmetric linear map $A\colon \mathbf{R}^n \to \mathbf{R}^n$, let $\sigma_j(A) \coloneqq \sigma_j(\kappa_1, \dots, \kappa_n)$ be the j^{th} elementary symmetric function of the eigenvalues κ_i of A, with $\sigma_0(A) \coloneqq 1$, and write $A_{ij} \coloneqq \langle Ae_i, e_j\rangle$. For $R \in \mathcal{R}_n$, define R^A via the Gauss equation

$$(R^A)_{ijk\ell} \coloneqq R_{ijk\ell} + A_{ik}A_{j\ell} - A_{i\ell}A_{jk}. \tag{3}$$

Equivalently, $R^A \coloneqq R + \frac{1}{2}A \owedge A$, where $\owedge$ is the Kulkarni–Nomizu product. Let

$$GK \coloneqq \det A$$

be the *Gauss–Kronecker curvature* of A. When A is diagonal, we set

$$L \coloneqq \sum_{i<j} K_{ij} \prod_{\ell \neq i,j} \kappa_\ell.$$

The next computation gives an algebraic decomposition of $\mathrm{Pf}(R^A)$, together with an estimate for its remainder, for use when the Gauss equation is substituted into the Chern–Gauss–Bonnet formulas below.

Lemma 2.3. *Let n be even, $A\colon \mathbf{R}^n \to \mathbf{R}^n$ be a diagonal linear map with entries $\kappa_1, \dots, \kappa_n$, and $R \in \mathcal{R}_n$. Then*

$$\mathrm{Pf}(R^A) = GK + \frac{1}{n-1}L + E_n.$$

If A is positive semidefinite, then

$$|E_n| \leq c_n \sum_{s=2}^{n/2} |R|^s \sigma_{n-2s}(A). \tag{4}$$

Furthermore, $E_4 = \mathrm{Pf}(R)$.

Proof. Let $Q \coloneqq \frac{1}{2}A \owedge A$, that is, $Q_{ijk\ell} = A_{ik}A_{j\ell} - A_{i\ell}A_{jk}$. So $R^A = R + Q$. By (2), $\mathrm{Pf}(R+Q) = \sum_{s=0}^{n/2} P_s$, where P_s is the sum of all terms containing exactly s factors of R and $n/2 - s$ factors of Q. Since A is diagonal, the nonzero components of Q are, up to symmetries, $Q_{ijij} = \kappa_i\kappa_j$. Hence in (2) each nonzero term of P_0 pairs the indices and contributes $\kappa_1 \cdots \kappa_n$, so $P_0 = \mathrm{Pf}(Q) = \det A = GK$.

Each term in P_1 contains one factor of R and $n/2 - 1$ factors of Q. Since the latter are nonzero only when their indices occur in equal pairs, the factor of R is of the form $R_{ijij} = K_{ij}$. Thus $P_1 = \mu_n L$. To determine μ_n, take $A = tI$ and $R = \tau R_1$. Then $R^A = (\tau + t^2)R_1$, and hence $\mathrm{Pf}(R^A) = (\tau + t^2)^{n/2}$. Comparing the terms linear in τ gives $\frac{n}{2}t^{n-2} = \mu_n\binom{n}{2}t^{n-2}$, so $\mu_n = 1/(n-1)$.

For $s \geq 2$, each term in P_s contains s factors of R and $n - 2s$ eigenvalues of A with distinct indices. Setting $E_n \coloneqq \sum_{s\geq 2} P_s$ gives the stated decomposition. If A is positive semidefinite, then $\kappa_\ell \geq 0$, and $|P_s| \leq c_n|R|^s\sigma_{n-2s}(A)$. Summing these estimates gives (4). Finally, when $n = 4$, the only term with $s \geq 2$ is $P_2 = \mathrm{Pf}(R)$, so $E_4 = \mathrm{Pf}(R)$. $\square$

2.2. **Estimates in dimension 4.** When $n = 4$, we have [6, (6.31)]

$$\mathrm{Pf}(R) = \frac{1}{24}\Big(|R|^2 - 4|\mathrm{Ric}_R|^2 + \mathrm{Scal}_R^2\Big). \tag{5}$$

To estimate this quantity, let $\Lambda^2\mathbf{R}^4$ be the space of 2-forms on $\mathbf{R}^4$, with the inner product $\langle x \wedge y, z \wedge w\rangle := \langle x, z\rangle\langle y, w\rangle - \langle x, w\rangle\langle y, z\rangle$, and norm $|\omega| := \langle \omega, \omega\rangle^{1/2}$. The *curvature operator* of $R \in \mathcal{R}_4$ is the self-adjoint map $\mathcal{R} \colon \Lambda^2\mathbf{R}^4 \to \Lambda^2\mathbf{R}^4$ given by

$$\langle \mathcal{R}(x \wedge y), z \wedge w\rangle := R(x, y, z, w).$$

Let $\Lambda^\pm \subset \Lambda^2\mathbf{R}^4$ be the eigenspaces, corresponding to the eigenvalues ± 1, of the *Hodge star operator*, i.e., the involution $* \colon \Lambda^2\mathbf{R}^4 \to \Lambda^2\mathbf{R}^4$ determined by $\alpha \wedge *\beta = \langle \alpha, \beta\rangle\, dV$, where $dV := e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Then $\Lambda^2\mathbf{R}^4 = \Lambda^+ \oplus \Lambda^-$, and elements of Λ^+, Λ^- are called *self-dual* and *anti-self-dual* forms respectively. With respect to this splitting, $\mathcal{R}$ assumes the block form

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}^+ & Z \\ Z^{\mathrm{T}} & \mathcal{R}^- \end{pmatrix},$$

where $|Z|^2 = \frac{1}{4}|\mathrm{Ric}_R - (\mathrm{Scal}_R/4)\,\langle\cdot,\cdot\rangle|^2$ [6]. A unit form $\omega \in \Lambda^2\mathbf{R}^4$ is *decomposable*, i.e., $\omega = x \wedge y$ for some vectors $x, y \in \mathbf{R}^4$ which may then be taken to be orthonormal, if and only if $\omega \wedge \omega = 0$. For such ω,

$$\langle \mathcal{R}\,\omega, \omega\rangle = R(x, y, x, y) = K(x, y).$$

Since $\omega \wedge \omega = \langle \omega, *\omega\rangle\, dV = \big(|\omega^+|^2 - |\omega^-|^2\big)dV$ for the components $\omega^\pm$ of ω in $\Lambda^\pm$, a unit form ω is decomposable if and only if $|\omega^+| = |\omega^-| = 1/\sqrt{2}$. Using these facts, we obtain the following estimate, which refines an earlier result of Ville [41]; see Note 2.5.

Lemma 2.4. *If $R \in \mathcal{R}_4$ with $-1 \leq K \leq 0$, then*

$$\mathrm{Pf}(R) \leq 1 - \Big(1 + \frac{\mathrm{Scal}_R}{12}\Big)^2 \leq 1.$$

Furthermore, $\mathrm{Pf}(R) = 1$ *if and only if* $R = R_{-1}$.

Proof. Since $|R|^2 = 4|\mathcal{R}|^2 = 4(|\mathcal{R}^+|^2 + |\mathcal{R}^-|^2 + 2|Z|^2)$, while $4|\mathrm{Ric}_R|^2 - \mathrm{Scal}_R^2 = 16|Z|^2$, (5) takes the form

$$\mathrm{Pf}(R) = \frac{1}{6}\big(|\mathcal{R}^+|^2 + |\mathcal{R}^-|^2\big) - \frac{1}{3}|Z|^2 \leq \frac{1}{6}\big(|\mathcal{R}^+|^2 + |\mathcal{R}^-|^2\big).$$

Note that $\mathrm{tr}\,\mathcal{R} = \sum_{i<j} R_{ijij} = \mathrm{Scal}_R/2$. Also, since $* = \pm 1$ on $\Lambda^\pm$,

$$\mathrm{tr}\,\mathcal{R}^+ - \mathrm{tr}\,\mathcal{R}^- = \mathrm{tr}(\mathcal{R} \circ *) = 2\big(R_{1234} + R_{1342} + R_{1423}\big) = 0,$$

where the middle equality is seen by computing the trace in the basis $e_i \wedge e_j$, and the last one is the first Bianchi identity. Hence $\mathrm{tr}\,\mathcal{R}^\pm = \frac{1}{2}\,\mathrm{tr}\,\mathcal{R} = \mathrm{Scal}_R/4 = 3t$, where $t := \mathrm{Scal}_R/12$. Let $\alpha_i^\pm$ be the eigenvalues of $\mathcal{R}^\pm$, with unit eigenvectors $\omega_i^\pm \in \Lambda^\pm$. The unit forms $(\omega_i^+ \pm \omega_j^-)/\sqrt{2}$ are decomposable, and the corresponding sectional curvatures are $(\alpha_i^+ + \alpha_j^- \pm 2\langle Z\omega_j^-, \omega_i^+\rangle)/2$. Their average is $(\alpha_i^+ + \alpha_j^-)/2$. Thus $w_{ij} := \alpha_i^+ + \alpha_j^- \in [-2, 0]$, which yields

$$\sum_{i,j=1}^{3} w_{ij}\big(w_{ij} + 2\big) \le 0.$$

Since $\sum_{ij} w_{ij} = 3\operatorname{tr}\mathcal{R}^+ + 3\operatorname{tr}\mathcal{R}^- = 18t$, and $\sum_{ij} w_{ij}^2 = 3\sum_i \big((\alpha_i^+)^2 + (\alpha_i^-)^2\big) + 2\operatorname{tr}\mathcal{R}^+\operatorname{tr}\mathcal{R}^-$, the last inequality may be rewritten as $\sum_i \big((\alpha_i^+)^2 + (\alpha_i^-)^2\big) \le -12t - 6t^2$. Hence

$$\operatorname{Pf}(R) \le \frac{1}{6}\sum_i \Big((\alpha_i^+)^2 + (\alpha_i^-)^2\Big) \le -2t - t^2 = 1 - (1+t)^2,$$

which establishes the desired estimate. Finally, if $\operatorname{Pf}(R) = 1$, then equality holds throughout: $Z = 0$, $t = -1$, and $w_{ij} \in \{0, -2\}$ for all i, j. Since $\sum_{ij} w_{ij} = 18t = -18$, it follows that $w_{ij} = -2$ for all i, j, which forces $\alpha_i^\pm = -1$ for all i. So $\mathcal{R} = -I$, or $R = R_{-1}$. Conversely, $|R_{-1}|^2 = 24$, $|\operatorname{Ric}_{R_{-1}}|^2 = 36$, and $\operatorname{Scal}_{R_{-1}} = -12$, so $\operatorname{Pf}(R_{-1}) = 1$ by (5). $\square$

Note 2.5. The inequality $\operatorname{Pf}(R) \le 1$ in Lemma 2.4, together with its equality case, is due to Ville [41, Thm. 1], who showed that the constant curvature tensor maximizes the Gauss–Bonnet integrand, which is the Pfaffian up to a normalizing factor. Integration via the Chern–Gauss–Bonnet theorem then gives $\chi(M) \le 3\operatorname{vol}(M)/(4\pi^2)$ for closed Riemannian 4-manifolds with $-1 \le K \le 0$, with equality if and only if $K \equiv -1$; see also Bettiol–Kummer–Mendes [7, Thm. E], where this inequality and related results are obtained via pointwise curvature estimates.

3. Chern–Gauss–Bonnet Defect Formulas

Here we use the Chern–Gauss–Bonnet theorem to compute the defect of the total curvature from the sharp bound in (1); see Proposition 3.2 below. To this end, let N be a compact oriented Riemannian n-manifold with smooth boundary ∂N, oriented by the outward unit normal ν. Let R and K denote the curvature tensor and sectional curvature of N, and let

$$A(X) := \nabla_X \nu$$

be the shape operator of ∂N. We recall the formulation of the Chern–Gauss–Bonnet theorem [11] given in [16, Sec. 2]. Let $e_1, \dots, e_{n-1}$ be a positively oriented local orthonormal frame on ∂N, and define

$$\alpha_i(X) := \langle A(X), e_i\rangle, \qquad \Omega_{ij}(X, Y) := R(X, Y, e_i, e_j).$$

If e_i is a principal direction of ∂N, then $\alpha_i(e_i) = \kappa_i$, where κ_i is the corresponding principal curvature. Thus the 1-forms α_i record the second fundamental form of ∂N. The 2-forms Ω_{ij}, which we call the *ambient curvature forms*, record the tangential components of the curvature of N. Suppose now that n is even. Then, for $0 \leq s \leq n/2-1$, the Chern $(n-1)$-forms are given by combining these forms as follows

$$\Phi_s := \sum_{\sigma\in S_{n-1}} \operatorname{sgn}(\sigma)\Big(\alpha_{\sigma(1)}\wedge\cdots\wedge\alpha_{\sigma(n-1-2s)}\Big) \wedge \Big(\Omega_{\sigma(n-2s)\sigma(n-2s+1)}\wedge\cdots\wedge\Omega_{\sigma(n-2)\sigma(n-1)}\Big),$$

where the second factor is omitted when $s=0$. The boundary term, which appears in the Chern–Gauss–Bonnet theorem for manifolds with boundary, is given by the transgression formula

$$\mathrm{TPf}_{\partial N} := \sum_{s=0}^{n/2-1} c_{n,s}\Phi_s(e_1,\dots,e_{n-1}), \tag{6}$$

where $c_{n,s} := |\mathbf{S}^{n-1}|/((4\pi)^s s!\,|\mathbf{S}^{n-1-2s}|(n-1-2s)!)$. These normalizing constants ensure that when the ambient curvature vanishes, $\mathrm{TPf}_{\partial N} = GK$. The Chern–Gauss–Bonnet theorem states that, when n is even,

$$|\mathbf{S}^{n-1}|\chi(N) = \frac{2|\mathbf{S}^{n-1}|}{|\mathbf{S}^n|}\int_N \mathrm{Pf}(R) + \int_{\partial N}\mathrm{TPf}_{\partial N}\,. \tag{7}$$

At each point of ∂N, choose the frame $e_1,\dots,e_{n-1}$ to be *principal*, i.e., to consist of principal directions, with $\kappa_1,\dots,\kappa_{n-1}$ the corresponding principal curvatures. Set $K_{ij} := K(e_i,e_j)$, and recall that $GK := \det A$ as in Section 2. The following result isolates the terms with at most one ambient curvature form in $\mathrm{TPf}_{\partial N}$ and gives an estimate for the remaining terms. For the terms with one curvature form, or the linear portion of TPf, recall that $L = \sum_{i<j} K_{ij}\prod_{\ell\neq i,j}\kappa_\ell$, as in Section 2.

Lemma 3.1. *Suppose that $n \geq 4$ is even. Then*

$$\mathrm{TPf}_{\partial N} = GK + \frac{1}{n-2}L + E_{n-1},$$

where E_{n-1} consists of the terms containing at least two ambient curvature forms. If A is positive semidefinite, then $|E_{n-1}| \leq c_n \sum_{s=2}^{n/2-1}|R|^s\sigma_{n-1-2s}(A)$.

Proof. Let $\theta^1,\dots,\theta^{n-1}$ be the coframe dual to $e_1,\dots,e_{n-1}$. Since the frame is principal, $\alpha_i = \kappa_i\theta^i$. Consequently,

$$c_{n,0}\Phi_0(e_1,\dots,e_{n-1}) = \kappa_1\cdots\kappa_{n-1} = GK,$$

and we obtain the first term in the stated formula. The term containing one curvature form is

$$c_{n,1}\Phi_1(e_1,\dots,e_{n-1}) = c_{n,1}2(n-3)!L.$$

Indeed, since $\alpha_i = \kappa_i\theta^i$, in the evaluation of each summand of Φ_1 on $(e_1,\dots,e_{n-1})$ the 1-forms must receive the vectors $e_{\sigma(1)},\dots,e_{\sigma(n-3)}$; so $\Omega_{\sigma(n-2)\sigma(n-1)}$ is evaluated on the remaining pair, which yields $K_{\sigma(n-2)\sigma(n-1)}$. Once the pair $\{i,j\}$ in Ω_{ij} is fixed, the remaining indices may be ordered in $(n-3)!$ ways, while the two orders of i, j give the same contribution. Since

$$c_{n,1} 2(n-3)! = \frac{|\mathbf{S}^{n-1}|}{2\pi|\mathbf{S}^{n-3}|} = \frac{1}{n-2},$$

we obtain the second term in $\mathrm{TPf}_{\partial N}$. Every remaining term contains $s \geq 2$ tangential components of R and $n-1-2s$ principal curvatures with distinct indices. If $\kappa_i \geq 0$, the same counting argument used to obtain (4) gives the stated estimate for E_{n-1}. □

The preceding computation and the Chern–Gauss–Bonnet theorem yield the following defect formulas. Set $\mathcal{G}(\partial N) \coloneqq \int_{\partial N} GK$.

Proposition 3.2 (Defect formulas)**.** *Suppose that $n \geq 3$. If n is even and N is contractible, then*

$$\mathcal{G}(\partial N) - |\mathbf{S}^{n-1}| = -\frac{2|\mathbf{S}^{n-1}|}{|\mathbf{S}^n|} \int_N \mathrm{Pf}(R) - \frac{1}{n-2} \int_{\partial N} L - \int_{\partial N} E_{n-1},$$

where E_{n-1} is as in Lemma 3.1; in particular $E_3 = 0$. If instead n is odd and ∂N is diffeomorphic to $\mathbf{S}^{n-1}$, then

$$\mathcal{G}(\partial N) - |\mathbf{S}^{n-1}| = -\frac{1}{n-2} \int_{\partial N} L - \int_{\partial N} E_{n-1},$$

where E_{n-1} is as in Lemma 2.3, applied to the restriction of R to $T\partial N$; in particular $E_4 = \mathrm{Pf}(R|_{T\partial N})$.

Proof. When n is even, $\chi(N) = 1$, since N is contractible; so integrating the formula of Lemma 3.1 over ∂N and substituting into (7) yields the first identity. When n is odd, ∂N has even dimension $n-1$ and $\chi(\mathbf{S}^{n-1}) = 2$; so (7) applied to ∂N gives $\int_{\partial N} \mathrm{Pf}(R^{\partial N}) = |\mathbf{S}^{n-1}|$, where $R^{\partial N}$ is the curvature tensor of ∂N. By the Gauss equation, $(R^{\partial N})_{ijk\ell} = R_{ijk\ell} + A_{ik}A_{j\ell} - A_{i\ell}A_{jk}$, so (3) shows that $R^{\partial N} = R^A$. Expanding $\mathrm{Pf}(R^A)$ by Lemma 2.3 in dimension $n-1$ then yields the second identity. □

4. Weighted Hsiung–Minkowski Inequalities

In this section, Γ is a smooth convex hypersurface in a Cartan–Hadamard manifold (M^n, g), bounding a convex body Ω. Let ν be the outward unit normal and A be the shape operator of Γ. For $0 \leq j \leq n-1$, let

$$\mathcal{M}_j = \mathcal{M}_j(\Gamma) \coloneqq \int_\Gamma \sigma_j(A)$$

be the j^{th} *total generalized mean curvature* of Γ. Thus $\mathcal{M}_0 = |\Gamma|$, $\mathcal{M}_1$ is the usual total mean curvature, and $\mathcal{M}_{n-1} = \mathcal{G}(\Gamma)$. Here we establish inequalities relating the successive total mean curvatures $\mathcal{M}_j$. Their common ingredients are Stokes theorem, Hessian comparison, and a weighted Hsiung–Minkowski identity obtained from Newton transformations. Applying this machinery to the squared distance from a point in Ω yields inequalities involving the diameter, while applying it to the distance from a point outside Ω yields inequalities independent of the diameter.

4.1. **Newton transformations.** The integral identities used below are rooted in the works of Hsiung [27,28] and Reilly [38]; see [1,20,32] and references therein for background and other recent applications. These identities generalize the classical Minkowski formulas for convex hypersurfaces in Euclidean space [34,39] and are naturally expressed in terms of Newton transformations. For $0 \leq j \leq n-1$, the j^{th} *Newton transformation* of A is defined recursively by

$$T_0 \coloneqq I, \qquad T_j \coloneqq \sigma_j(A)I - AT_{j-1}.$$

Since A is a smooth self-adjoint $(1,1)$-tensor field on Γ, so are T_j.

Lemma 4.1. *For* $1 \leq j \leq n-1$,

$$\operatorname{tr} T_{j-1} = (n-j)\sigma_{j-1}(A), \qquad \operatorname{tr}(AT_{j-1}) = j\sigma_j(A). \tag{8}$$

If A is positive semidefinite, then $0 \leq T_{j-1} \leq \sigma_{j-1}(A)I$.

Proof. In a principal frame e_i, with principal curvatures κ_i,

$$T_{j-1}e_i = \sigma_{j-1}(\kappa_1, \ldots, \widehat{\kappa_i}, \ldots, \kappa_{n-1})e_i,$$

which follows by induction on j, since

$$\sigma_j(A) - \kappa_i\, \sigma_{j-1}(\kappa_1, \ldots, \widehat{\kappa_i}, \ldots, \kappa_{n-1}) = \sigma_j(\kappa_1, \ldots, \widehat{\kappa_i}, \ldots, \kappa_{n-1}).$$

Each monomial in $\sigma_{j-1}(A)$ occurs in $n-j$ of these eigenvalues, while each monomial in $\sigma_j(A)$ occurs j times after multiplication by the corresponding principal curvature. This proves (8). If A is positive semidefinite, each eigenvalue of T_{j-1} lies between 0 and $\sigma_{j-1}(A)$, which proves the final assertion. □

The Hsiung–Minkowski formulas [27,33] for a smooth closed hypersurface $\Gamma \subset \mathbf{R}^n$ state that, for $1 \leq j \leq n-1$,

$$j \int_\Gamma \langle x, \nu \rangle \sigma_j(A) = (n-j) \int_\Gamma \sigma_{j-1}(A),$$

where $\langle x, \nu \rangle$ is the support function of Γ. For $j = 1$, they reduce to the classical Minkowski formula $\int_\Gamma \langle x, \nu \rangle \sigma_1(A) = (n-1)|\Gamma|$, [34, Thm. 6.11]; see [39, Eq. (5.60)]. The following proposition gives a weighted generalization of these formulas. Indeed, in Euclidean space, choosing $f(x) \coloneqq |x|^2/2$ in the proposition recovers the formulas above, since the gradient $\nabla f = x$, the Hessian $\nabla^2 f$ restricts to the identity on $T\Gamma$, and the Newton transformations are divergence-free. For a $(1,1)$-tensor field T on Γ, we write

$$\operatorname{div} T \coloneqq \sum_{i=1}^{n-1} \left(\nabla^\Gamma_{e_i} T \right) e_i,$$

where ∇^Γ denotes the induced connection on Γ, and $e_1, \ldots, e_{n-1}$ is a local orthonormal frame on Γ; the divergence of tangent vector fields on Γ is defined similarly. Furthermore, $\nabla_\Gamma f$ and $\nabla^2_\Gamma f$ denote the gradient and Hessian of the restriction of f

to Γ, computed with respect to the induced metric and connection of Γ; in particular, $\nabla_\Gamma f$ is the tangential component of ∇f.

Proposition 4.2 (Weighted Hsiung–Minkowski identity)**.** *Let f be a smooth function in a neighborhood of Γ. Then, for $1 \leq j \leq n-1$,*

$$j\int_\Gamma \langle \nabla f, \nu\rangle \sigma_j(A) = \int_\Gamma \operatorname{tr}\left(T_{j-1}\nabla^2 f|_{T\Gamma}\right) + \int_\Gamma \langle \operatorname{div} T_{j-1}, \nabla_\Gamma f\rangle \,. \tag{9}$$

Proof. On Γ, we have $\nabla f = \nabla_\Gamma f + \langle \nabla f, \nu\rangle\nu$. So, for X, Y tangent to Γ,

$$\nabla^2 f(X,Y) = \langle \nabla_X \nabla_\Gamma f, Y\rangle + \langle \nabla f, \nu\rangle\langle \nabla_X \nu, Y\rangle = \nabla^2_\Gamma f(X,Y) + \langle \nabla f, \nu\rangle A(X,Y),$$

since $\nabla^2 f(X,Y) = \langle \nabla_X \nabla f, Y\rangle$, and $\nabla^2_\Gamma f(X,Y) = \langle \nabla_X \nabla_\Gamma f, Y\rangle$. Thus we obtain

$$\nabla^2_\Gamma f = \nabla^2 f|_{T\Gamma} - \langle \nabla f, \nu\rangle A.$$

Since T_{j-1} is self-adjoint, $\operatorname{div}\left(T_{j-1}\nabla_\Gamma f\right) = \langle \operatorname{div} T_{j-1}, \nabla_\Gamma f\rangle + \operatorname{tr}\left(T_{j-1}\nabla^2_\Gamma f\right)$ by the product rule [2, Eq. (8.2)]. As Γ is closed, the Stokes theorem gives

$$0 = \int_\Gamma \operatorname{div}\left(T_{j-1}\nabla_\Gamma f\right) = \int_\Gamma \operatorname{tr}(T_{j-1}\nabla^2_\Gamma f) + \int_\Gamma \langle \operatorname{div} T_{j-1}, \nabla_\Gamma f\rangle.$$

Substitution of the preceding Hessian identity and (8) proves (9). $\square$

4.2. **Divergence estimates.** The last term in (9) is absent from the classical Hsiung–Minkowski formulas. To derive inequalities from (9), we therefore need to bound $\operatorname{div} T_{j-1}$ in terms of the ambient curvature and lower-order mean curvatures. The next lemma provides this bound, together with the sharper estimate for T_1 needed under curvature pinching. Here we regard Ric_R as a self-adjoint operator on $\mathbf{R}^n$, given by $\langle \operatorname{Ric}_R(x), y\rangle \coloneqq \operatorname{Ric}_R(x,y)$. Let $(\cdot)^\top$ be the projection onto $T\Gamma$.

Lemma 4.3. *We have $\operatorname{div} T_0 = 0$, and, for $2 \leq j \leq n-1$,*

$$|\operatorname{div} T_{j-1}| \leq c_n |R| \sigma_{j-2}(A).$$

Furthermore, $|\operatorname{div} T_1| = |(\operatorname{Ric}_R(\nu))^\top|$. In particular, if $-1 \leq K \leq -\lambda$, then

$$|\operatorname{div} T_1| \leq \frac{n-2}{2}(1-\lambda). \tag{10}$$

Proof. Since $T_0 = I$, its divergence vanishes. The Newton transformations can be written as

$$(T_{j-1})_{ik} = \frac{1}{(j-1)!}\sum \delta^{k\,a_1\cdots a_{j-1}}_{i\,b_1\cdots b_{j-1}} A_{a_1 b_1}\cdots A_{a_{j-1}b_{j-1}},$$

where $\delta^{i_1\cdots i_k}_{j_1\cdots j_k}$ is the generalized Kronecker delta, and the sum ranges over indices from 1 to $n-1$ [2,38]. Differentiating in an orthonormal frame which is parallel at a given point, and using the symmetry of the summands in the pairs (a_q, b_q), we obtain

$$(\operatorname{div} T_{j-1})_i = \frac{1}{(j-2)!}\sum \delta^{k\,a_1\cdots a_{j-1}}_{i\,b_1\cdots b_{j-1}} (\nabla_{e_k} A)_{a_1 b_1} A_{a_2 b_2}\cdots A_{a_{j-1}b_{j-1}}. \tag{11}$$

Since the Kronecker delta is antisymmetric in k, a_1, only the part of $(\nabla_{e_k} A)_{a_1 b_1}$ antisymmetric in these indices contributes. The Codazzi equation gives

$$(\nabla_{e_k} A)_{a_1 b_1} - (\nabla_{e_{a_1}} A)_{k b_1} = R(e_k, e_{a_1}, e_{b_1}, \nu). \tag{12}$$

Thus this antisymmetric part is given by components of R. Choosing the frame principal at the given point, each nonzero summand in (11) is bounded in absolute value by $|R|$ times a product of $j-2$ principal curvatures with distinct indices, by the alternating property of the delta. Summing these terms and using $\kappa_i \geq 0$ gives the first stated estimate.

For $j = 2$, (11) reduces to $\operatorname{div} T_1 = \nabla_\Gamma \sigma_1(A) - \operatorname{div} A$. On the other hand, setting $b_1 = k$ in (12) and summing over k yields the contracted Codazzi equation

$$\operatorname{div} A - \nabla_\Gamma \sigma_1(A) = \big(\operatorname{Ric}_R(\nu)\big)^\top.$$

Hence $\operatorname{div} T_1 = -(\operatorname{Ric}_R(\nu))^\top$. Now suppose that $-1 \leq K \leq -\lambda$, and write $R = R_{-1} + S$. Then the sectional curvature of S lies between 0 and $1-\lambda$. Furthermore $\operatorname{Ric}_{R_{-1}} = -(n-1)\langle\cdot,\cdot\rangle$, so the mixed components $\operatorname{Ric}_{R_{-1}}(\nu, X)$, for X tangent to Γ, vanish; hence $\operatorname{Ric}_R(\nu, X) = \operatorname{Ric}_S(\nu, X)$. For a unit tangent vector X, complete X, ν to an orthonormal frame $X, e_2, \ldots, e_{n-1}, \nu$. Lemma 2.2 gives

$$|\operatorname{Ric}_R(\nu, X)| = \left|\sum_{i=2}^{n-1} S(e_i, \nu, e_i, X)\right| \leq \frac{n-2}{2}(1-\lambda),$$

which proves (10). $\square$

4.3. **Inequalities involving diameter.** We first derive inequalities involving the diameter of Γ by applying the weighted identity (9) to $f := r^2/2$, where r denotes distance from a point chosen in the interior of Ω. For this choice, f has uniformly positive Hessian, while $|\nabla f| = r$ is bounded by the diameter. The resulting comparisons between successive total mean curvatures will be used to control the terms containing at least two ambient curvature factors in the proof of Theorem 1.1.

Lemma 4.4. *Suppose that $|R| \leq B^2$ on Ω. Then, for $1 \leq j \leq n-1$,*

$$(n-j)\mathcal{M}_{j-1}(\Gamma) \leq \Big(j\mathcal{M}_j(\Gamma) + c_n B^2 \mathcal{M}_{j-2}(\Gamma)\Big)\operatorname{diam}(\Gamma), \tag{13}$$

where the final term is omitted when $j = 1$. Furthermore,

$$|\Omega| \leq \frac{\operatorname{diam}(\Gamma)}{n}|\Gamma|, \tag{14}$$

independently of B.

Proof. Choose a point p in the interior of Ω, let $r := \operatorname{dist}_M(p, \cdot)$, and set $f := r^2/2$. Since M is Cartan–Hadamard, Hessian comparison gives $\nabla^2 f \geq g$ [37, Lem. 6.2.5]. Put $d := \operatorname{diam}(\Gamma)$. Convexity gives $\operatorname{diam}(\Omega) = d$, and hence $|\nabla f| = r \leq d$ on Ω. Since $T_{j-1} \geq 0$, Proposition 4.2 and Lemmas 4.1 and 4.3 give

$$jd\mathcal{M}_j \geq j\int_\Gamma \langle \nabla f, \nu\rangle \sigma_j(A) \geq (n-j)\mathcal{M}_{j-1} - c_n B^2 d\mathcal{M}_{j-2},$$

where the final term is absent when $j = 1$, since $\operatorname{div} T_0 = 0$. This proves (13). Furthermore, $\Delta f \geq n$, so the Stokes theorem gives $n|\Omega| \leq \int_\Gamma \langle \nabla f, \nu \rangle$. Since $\langle \nabla f, \nu \rangle \leq |\nabla f| \leq d$, we obtain (14). □

Proposition 4.5. *For $n \geq 3$, under the hypotheses of Lemma 4.4, there exists $\varepsilon_n > 0$ such that, if $B \operatorname{diam}(\Gamma) \leq \varepsilon_n$, then for $2 \leq s \leq \lfloor (n-1)/2 \rfloor$,*

$$\mathcal{M}_{n-1-2s}(\Gamma) \leq c_n \operatorname{diam}(\Gamma)^{2s-2} \mathcal{M}_{n-3}(\Gamma),$$

and

$$|\Omega| \leq c_n \operatorname{diam}(\Gamma)^{n-2} \mathcal{M}_{n-3}(\Gamma).$$

Proof. Put $d := \operatorname{diam}(\Gamma)$. If $n = 3$, the result follows directly from (14). So suppose that $n \geq 4$. We first show that $\mathcal{M}_{j-1} \leq c_n d \mathcal{M}_j$, $1 \leq j \leq n-3$. For $j = 1$, (13) gives $|\Gamma| \leq c_n d \mathcal{M}_1$. Suppose inductively that $\mathcal{M}_{j-2} \leq c_n d \mathcal{M}_{j-1}$. Then (13) yields

$$\mathcal{M}_{j-1} \leq c_n d \mathcal{M}_j + c_n (Bd)^2 \mathcal{M}_{j-1}.$$

Choosing ε_n sufficiently small and absorbing the final term proves the claim. Iterating the claim from $\mathcal{M}_{n-1-2s}$ to $\mathcal{M}_{n-3}$ gives $\mathcal{M}_{n-1-2s} \leq c_n d^{2s-2} \mathcal{M}_{n-3}$. Similarly, $|\Gamma| \leq c_n d^{n-3} \mathcal{M}_{n-3}$. Combining this with (14) gives $|\Omega| \leq c_n d |\Gamma| \leq c_n d^{n-2} \mathcal{M}_{n-3}$. □

4.4. **Inequalities independent of diameter.** We next derive inequalities independent of the diameter of Γ. Fix $p \notin \Omega$, set $r(\cdot) := \operatorname{dist}_M(p, \cdot)$, and apply the weighted identity (9) to $f := r$. For this choice, Hessian comparison provides a lower bound for $\nabla^2 r$ independent of the size of Ω. The resulting inequalities will be used in dimensions 4 and 5.

Lemma 4.6. *Suppose that $K \leq -a^2 < 0$. Then, for $1 \leq j \leq n-1$,*

$$j \int_\Gamma \langle \nabla r, \nu \rangle \sigma_j(A) \geq a \int_\Gamma \big(n - 1 - j + \langle \nabla r, \nu \rangle^2\big) \sigma_{j-1}(A) - \int_\Gamma |\operatorname{div} T_{j-1}|. \tag{15}$$

Moreover,

$$\int_\Gamma \langle \nabla r, \nu \rangle \geq (n-1) a |\Omega|. \tag{16}$$

Proof. Since M is Cartan–Hadamard and $p \notin \Omega$, the function r is smooth on Ω, and Hessian comparison gives

$$\nabla^2 r(X, X) \geq a \coth(ar) \big(|X|^2 - \langle \nabla r, X \rangle^2\big) \geq a \big(|X|^2 - \langle \nabla r, X \rangle^2\big), \tag{17}$$

see [37, Thm. 6.4.3]. Since $|\nabla r| = 1$ and $\nabla r = \nabla_\Gamma r + \langle \nabla r, \nu \rangle \nu$ along Γ, we have

$$|\nabla_\Gamma r|^2 = 1 - \langle \nabla r, \nu \rangle^2.$$

Let $e_1, \dots, e_{n-1}$ be an orthonormal basis of eigenvectors of T_{j-1} on $T\Gamma$, with corresponding eigenvalues τ_i, which are nonnegative by Lemma 4.1. By (17),

$$\tau_i \nabla^2 r(e_i, e_i) \geq a \tau_i \big(1 - \langle \nabla r, e_i \rangle^2\big) = a \tau_i \big(1 - \langle \nabla_\Gamma r, e_i \rangle^2\big),$$

where the equality holds since e_i are tangent to Γ. Summing over i then yields

$$\operatorname{tr}\left(T_{j-1}\nabla^2 r|_{T\Gamma}\right) = \sum_{i=1}^{n-1} \tau_i\,\nabla^2 r(e_i,e_i) \geq a\Big(\operatorname{tr} T_{j-1} - T_{j-1}\big(\nabla_\Gamma r, \nabla_\Gamma r\big)\Big).$$

Lemma 4.1 gives $\operatorname{tr} T_{j-1} = (n-j)\sigma_{j-1}(A)$, and $T_{j-1} \leq \sigma_{j-1}(A) I$, which in turn yields $T_{j-1}(\nabla_\Gamma r, \nabla_\Gamma r) \leq \sigma_{j-1}(A)|\nabla_\Gamma r|^2$. So

$$\operatorname{tr}\left(T_{j-1}\nabla^2 r|_{T\Gamma}\right) \geq a\big(n-j-|\nabla_\Gamma r|^2\big)\sigma_{j-1}(A) = a\big(n-1-j+\langle\nabla r,\nu\rangle^2\big)\sigma_{j-1}(A).$$

Substituting this estimate into (9), with $f := r$, and noting that $\left|\left\langle \operatorname{div} T_{j-1}, \nabla_\Gamma r\right\rangle\right| \leq |\operatorname{div} T_{j-1}|$, since $|\nabla_\Gamma r| \leq 1$, yields (15). Finally, taking the trace in (17) gives $\Delta r \geq (n-1)a$. So the Stokes theorem yields

$$\int_\Gamma \langle \nabla r, \nu\rangle = \int_\Omega \Delta r \geq (n-1)a|\Omega|,$$

which is (16). □

Using the first two cases of (15), for $j=1$ and $j=2$, we now obtain:

Proposition 4.7. *Under the hypotheses of Lemma 4.6,*

$$\mathcal{M}_1(\Gamma) \geq (n-1)^2 a^2 |\Omega|, \tag{18}$$

for $n \geq 3$. If $n \geq 4$ and $-1 \leq K \leq -\lambda < 0$, then

$$2\mathcal{M}_2(\Gamma) \geq \left((n-2)^2\lambda - \frac{n-2}{2}(1-\lambda)\right)|\Gamma|. \tag{19}$$

Proof. Set $U_0 := \int_\Gamma \langle \nabla r, \nu\rangle$. Since $\sigma_0(A) = 1$, and $T_0 = I$ is divergence free, the case $j=1$ of (15) reads

$$\int_\Gamma \langle \nabla r, \nu\rangle \sigma_1(A) \geq a\left((n-2)|\Gamma| + \int_\Gamma \langle \nabla r, \nu\rangle^2\right).$$

Since $\langle \nabla r, \nu\rangle \leq 1$ and $\sigma_1(A) \geq 0$, the left-hand side is at most $\mathcal{M}_1$, while $\int_\Gamma \langle\nabla r,\nu\rangle^2 \geq U_0^2/|\Gamma|$ by the Cauchy–Schwarz inequality. Hence

$$\mathcal{M}_1 \geq a\left((n-2)|\Gamma| + \frac{U_0^2}{|\Gamma|}\right) \geq (n-1)aU_0,$$

where the last inequality holds since

$$(n-2)|\Gamma|^2 + U_0^2 - (n-1)U_0|\Gamma| = \big((n-2)|\Gamma| - U_0\big)\big(|\Gamma| - U_0\big) \geq 0,$$

because $U_0 \leq |\Gamma|$. Together with (16), this proves (18).

Next, to obtain (19), set $U_1 := \int_\Gamma \langle\nabla r,\nu\rangle\sigma_1(A)$. The first display above gives $U_1 \geq a(n-2)|\Gamma|$. Furthermore, the case $j=2$ of (15), together with $\langle\nabla r,\nu\rangle \leq 1$, $\sigma_2(A) \geq 0$, and the Cauchy–Schwarz inequality $\int_\Gamma \langle\nabla r,\nu\rangle^2\sigma_1(A) \geq U_1^2/\mathcal{M}_1$, yields

$$2\mathcal{M}_2 \geq a\left((n-3)\mathcal{M}_1 + \frac{U_1^2}{\mathcal{M}_1}\right) - \int_\Gamma |\operatorname{div} T_1| \geq a(n-2)U_1 - \int_\Gamma |\operatorname{div} T_1|,$$

where the last inequality holds, as above, since

$$(n-3)\mathcal{M}_1^2 + U_1^2 - (n-2)U_1\mathcal{M}_1 = \big((n-3)\mathcal{M}_1 - U_1\big)\big(\mathcal{M}_1 - U_1\big) \geq 0,$$

because $n \geq 4$ and $U_1 \leq \mathcal{M}_1$. Consequently,

$$2\mathcal{M}_2 \geq a^2(n-2)^2|\Gamma| - \int_\Gamma |\operatorname{div} T_1|.$$

If $-1 \leq K \leq -\lambda$, then we may take $a = \sqrt{\lambda}$, and (10) yields (19). $\square$

Note 4.8. Since $\langle \nabla r, \nu\rangle \leq 1$, (16) yields the linear isoperimetric inequality $|\Gamma| \geq (n-1)a|\Omega|$, which holds for domains in Cartan–Hadamard manifolds with $K \leq -a^2$. This inequality was obtained by Yau [44], whose proof also rests on applying the divergence theorem to ∇r. See Burago–Zalgaller [9, Thm. 34.2.6] for another proof, via volume comparison, and [9, Rem. 34.2.8], [26] for quantitative refinements.

5. Convex Hypersurfaces of Small Diameter

Here we prove Theorem 1.1. The argument has two ingredients: the Chern–Gauss–Bonnet expansion isolates a favorable term involving $\mathcal{M}_{n-3}$, while the estimates of Section 4.3 control all lower-order terms. The Chern–Gauss–Bonnet input, depending on the parity of n, is supplied by Proposition 3.2. First we establish a stronger quantitative result.

Proposition 5.1. *For every $n \geq 4$ there exist constants $\varepsilon_n, c_n > 0$ with the following property. Let M^n be a Cartan–Hadamard manifold, $\Gamma \subset M$ a smooth convex hypersurface bounding the convex body Ω, and suppose that, on Ω, $K \leq -a^2$ and $|R| \leq B^2$ for some $a, B > 0$. If $B \operatorname{diam}(\Gamma) \leq \varepsilon_n$, then*

$$\mathcal{G}(\Gamma) - |\mathbf{S}^{n-1}| \geq \left(\frac{a^2}{n-2} - c_n B^4 \operatorname{diam}(\Gamma)^2\right)\mathcal{M}_{n-3}(\Gamma).$$

Note that if $K \leq -a^2$ on Γ, then, since $\kappa_i \geq 0$,

$$L \leq -a^2\sigma_{n-3}(A) \tag{20}$$

at each point of Γ. We also need the following estimate:

Lemma 5.2. *Let M, Γ, Ω, a, and B be as in Proposition 5.1. Then*

$$\mathcal{G}(\Gamma) - |\mathbf{S}^{n-1}| \geq \frac{a^2}{n-2}\mathcal{M}_{n-3}(\Gamma) - c_n \sum_{s=2}^{\lfloor (n-1)/2 \rfloor} B^{2s}\mathcal{M}_{n-1-2s}(\Gamma) - c_n B^n|\Omega|,$$

where the final term is omitted when n is odd.

Proof. If n is odd, then Γ, which bounds a convex body, is diffeomorphic to $\mathbf{S}^{n-1}$, and the odd case of Proposition 3.2 applies. If n is even, then Ω is contractible, and the even case of Proposition 3.2 applies, where the homogeneity of the Pfaffian gives $|\operatorname{Pf}(R)| \leq c_n B^n$. In either case, since $|R| \leq B^2$, the bounds on E_{n-1} in Lemmas 2.3 and 3.1 yield $|E_{n-1}| \leq c_n \sum_{s=2}^{\lfloor (n-1)/2\rfloor} B^{2s}\sigma_{n-1-2s}(A)$. The result follows by integrating these estimates together with (20). $\square$

Proof of Proposition 5.1. Put $d := \operatorname{diam}(\Gamma)$, and shrink ε_n if necessary so that $\varepsilon_n \leq 1$. By Proposition 4.5,

$$B^{2s}\mathcal{M}_{n-1-2s} \leq c_n B^4 d^2 \mathcal{M}_{n-3}, \qquad B^n|\Omega| \leq c_n B^4 d^2 \mathcal{M}_{n-3},$$

where the first estimate holds for $s \geq 2$, and the second is needed only when n is even. The result follows from Lemma 5.2. □

Proof of Theorem 1.1. The pinching assumption implies $|R| \leq c_n b^2$ by Lemma 2.1, after scaling. Thus Proposition 5.1 applies with $B = \sqrt{c_n}\, b$, which, after renaming ε_n and c_n, gives

$$\mathcal{G}(\Gamma) - |\mathbf{S}^{n-1}| \geq \left(\frac{a^2}{n-2} - c_n b^4 \operatorname{diam}(\Gamma)^2\right)\mathcal{M}_{n-3}(\Gamma),$$

provided that $b \operatorname{diam}(\Gamma)$ is sufficiently small. If $d := \operatorname{diam}(\Gamma) \leq \delta_n a/b^2$, then $bd \leq \delta_n a/b \leq \delta_n$ and $b^4 d^2 \leq \delta_n^2 a^2$. Choosing δ_n sufficiently small therefore ensures both the hypothesis of Proposition 5.1 and the nonnegativity of the coefficient above. Since $\mathcal{M}_{n-3} \geq 0$, the result follows. □

Note 5.3. Choosing δ_n so that $c_n\delta_n^2 \leq 1/(2(n-2))$ in the above proof yields the quantitative estimate

$$\mathcal{G}(\Gamma) - |\mathbf{S}^{n-1}| \geq \frac{a^2}{2(n-2)}\,\mathcal{M}_{n-3}(\Gamma).$$

Diameter-free defect bounds in dimensions 4 and 5 are given below in Propositions 6.1 and 6.3.

6. Diameter-Free Estimates in Low Dimensions

Here we prove Theorem 1.2. In both dimensions 4 and 5 the Chern–Gauss–Bonnet formula reduces the desired inequality to a comparison between lower-order geometric quantities, which is handled by estimates of Section 4.4. More specifically, after replacing the metric g of M by $b^2 g$, the curvature satisfies $-1 \leq K \leq -\lambda_n$. Since $\mathcal{G}(\Gamma)$ is invariant under constant rescaling, the result follows immediately from Propositions 6.1 and 6.3 below.

6.1. Dimension 4.

Here the only unfavorable term in the Chern–Gauss–Bonnet formula is the interior Pfaffian integral. We control it using the comparison between $\mathcal{M}_1$ and $|\Omega|$ obtained in (18).

Proposition 6.1. *Let M^4 be a Cartan–Hadamard manifold with $-1 \leq K \leq -\lambda$, where $\lambda \geq \lambda_4$. Then every smooth convex hypersurface $\Gamma \subset M$ satisfies*

$$\mathcal{G}(\Gamma) - |\mathbf{S}^3| \geq \frac{3\lambda^2 - 1}{6\lambda}\,\mathcal{M}_1(\Gamma) \geq 0.$$

Proof. Let Ω be the convex body bounded by Γ. Then Ω is contractible. So the even case of Proposition 3.2, where $E_3 = 0$ and $2|\mathbf{S}^3|/|\mathbf{S}^4| = 3/2$, together with (20) for $a = \sqrt{\lambda}$, gives

$$\mathcal{G}(\Gamma) - |\mathbf{S}^3| \geq \frac{\lambda}{2}\mathcal{M}_1(\Gamma) - \frac{3}{2}\int_\Omega \operatorname{Pf}(R).$$

By Lemma 2.4, $\int_\Omega \mathrm{Pf}(R) \leq |\Omega|$. On the other hand, (18), with $n=4$ and $a=\sqrt{\lambda}$, gives $\mathcal{M}_1 \geq 9\lambda|\Omega|$. Consequently,

$$\mathcal{G}(\Gamma) - |\mathbf{S}^3| \geq \left(\frac{\lambda}{2} - \frac{1}{6\lambda}\right)\mathcal{M}_1(\Gamma) = \frac{3\lambda^2-1}{6\lambda}\,\mathcal{M}_1(\Gamma).$$

The coefficient is nonnegative for $\lambda_4 \leq \lambda \leq 1$. $\square$

Note 6.2. The constant $\lambda_4 = 1/\sqrt{3}$ results from combining two estimates which are sharp separately. Equality in Lemma 2.4 requires $R = R_{-1}$, whereas (18) is asymptotically sharp for large geodesic spheres in the space form of curvature $-\lambda$. These sharpness mechanisms are different when $\lambda < 1$. In particular, the stronger estimate $\mathrm{Pf}(R) \leq 1 - (1 + \mathrm{Scal}_R/12)^2$ from Lemma 2.4 has not been used, nor has any relation between $\int_\Omega \mathrm{Pf}(R)$ and $\mathcal{M}_1(\Gamma)$. Exploiting such information could potentially improve λ_4.

6.2. **Dimension 5.** Here the intrinsic Chern–Gauss–Bonnet formula reduces the problem to comparing $\mathcal{M}_2$ and $|\Gamma|$. The required comparison is supplied by (19).

Proposition 6.3. *Let M^5 be a Cartan–Hadamard manifold with $-1 \leq K \leq -\lambda$, where $\lambda \geq \lambda_5$. Then every smooth convex hypersurface $\Gamma \subset M$ satisfies*

$$\mathcal{G}(\Gamma) - |\mathbf{S}^4| \geq \frac{7\lambda^2 - \lambda - 4}{4}\,|\Gamma| \geq 0.$$

Proof. Since Γ bounds a convex body, it is diffeomorphic to $\mathbf{S}^4$. So the case $n=5$ of Proposition 3.2 gives

$$\mathcal{G}(\Gamma) - |\mathbf{S}^4| = -\frac{1}{3}\int_\Gamma L - \int_\Gamma \mathrm{Pf}(R|_{T\Gamma}).$$

By (20), with $a = \sqrt{\lambda}$, and Lemma 2.4, $\mathcal{G}(\Gamma) - |\mathbf{S}^4| \geq \frac{\lambda}{3}\mathcal{M}_2 - |\Gamma|$. Formula (19), with $n=5$, gives $2\mathcal{M}_2 \geq (21\lambda - 3)|\Gamma|/2$. Therefore

$$\mathcal{G}(\Gamma) - |\mathbf{S}^4| \geq \left(\frac{\lambda(21\lambda-3)}{12} - 1\right)|\Gamma| = \frac{7\lambda^2-\lambda-4}{4}\,|\Gamma|.$$

The coefficient is nonnegative for $\lambda_5 \leq \lambda \leq 1$. $\square$

Note 6.4. Eliminating the divergence term in the proof of (19) would replace the coefficient in Proposition 6.3 by $3\lambda^2/2 - 1$, lowering the required pinching only to $\lambda \geq \sqrt{2/3}$. On the other hand, (19) is not asymptotically sharp: for large geodesic spheres in the space form of curvature $-\lambda$, $2\mathcal{M}_2(\Gamma)/|\Gamma| \to (n-1)(n-2)\lambda$. Thus λ_5 may potentially be improved by retaining information discarded in the proof of Proposition 4.7, such as the term $\int_\Gamma \langle \nabla r, \nu\rangle^2$.

7. Further Notes

Note 7.1. The pinching hypotheses need only hold near Ω in the above results. In Theorem 1.1 the curvature bounds are required only on Ω, by Proposition 5.1. In Theorem 1.2 the exterior center p may be chosen arbitrarily close to Γ; since the distance between geodesic segments with a common endpoint is a convex function, the segments joining p to Ω then lie in any prescribed neighborhood of Ω, which is all that the proofs of the exterior radial estimates require.

Note 7.2. The smoothness assumption on Γ in Theorems 1.1 and 1.2 may be removed. For an arbitrary convex hypersurface $\Gamma \subset M^n$, its outer parallel hypersurface Γ^ε, at distance $\varepsilon > 0$, is $\mathcal{C}^{1,1}$ [18]. Thus $\mathcal{G}(\Gamma^\varepsilon)$ is well-defined by Rademacher's theorem, and we set

$$\mathcal{G}(\Gamma) := \lim_{\varepsilon \searrow 0} \mathcal{G}(\Gamma^\varepsilon).$$

This limit exists since $\varepsilon \mapsto \mathcal{G}(\Gamma^\varepsilon)$ is nondecreasing. It also agrees with the usual definition when Γ is smooth, since $\mathcal{G}$ is continuous with respect to Hausdorff distance [15, Thm. 1.2]. By the Greene–Wu smoothing procedure [23], as carried out in [17, Lem. 3.9], there exist smooth convex bodies $\Omega_i \subset \Omega$ which converge to Ω in the Hausdorff metric. Set $\Gamma_i := \partial\Omega_i$. Then $\operatorname{diam}(\Gamma_i) \le \operatorname{diam}(\Gamma)$, the ambient curvature hypotheses hold on Ω_i, and $\mathcal{G}(\Gamma_i) \to \mathcal{G}(\Gamma)$. Applying the theorems to Γ_i and passing to the limit yields the same conclusions for arbitrary convex hypersurfaces.

Note 7.3. The diameter-free argument for pinched variable curvature, used in the proof of Theorem 1.2, already encounters an obstruction in dimension 6. Suppose that $-1 \le K \le -\lambda < 0$. By Lemma 2.1, multilinearity of the Pfaffian, and the transgression formula (6),

$$|\operatorname{Pf}(R) + 1| \le c_6(1-\lambda), \qquad \left| E_5 - \frac{3}{8}\sigma_1(A) \right| \le c_6(1-\lambda)\sigma_1(A).$$

Unlike in dimension 4, nonpositive sectional curvature alone does not determine the sign of the Pfaffian in dimension 6. Indeed, Geroch [14] constructed a 6-dimensional curvature tensor with positive sectional curvatures and negative Pfaffian. Since the Pfaffian is cubic in R, replacing R by $-R$ gives negative sectional curvatures and positive Pfaffian. Furthermore, $K_{ij} \le -\lambda$ gives $-L \ge \lambda\sigma_3(A)$. Hence Proposition 3.2 yields

$$\mathcal{G}(\Gamma) - |\mathbf{S}^5| \ge \left(\frac{15}{8} - c_6(1-\lambda)\right)|\Omega| + \frac{\lambda}{4}\mathcal{M}_3 - \left(\frac{3}{8} + c_6(1-\lambda)\right)\mathcal{M}_1.$$

The case $j = 1$ of (15) gives $\mathcal{M}_1 \ge 4\sqrt{\lambda}\,|\Gamma|$. Using (10), the case $j = 2$ gives

$$\int_\Gamma \langle \nabla r, \nu\rangle \sigma_2(A) \ge \left(\frac{3}{2}\sqrt{\lambda} - \frac{1-\lambda}{4\sqrt{\lambda}}\right)\mathcal{M}_1.$$

For the case $j = 3$, note that $2 + \langle \nabla r, \nu\rangle^2 \ge 3\langle \nabla r, \nu\rangle$. By (11) and (12), $\operatorname{div} T_2$ involves only curvature components with one normal and three tangential entries. These components vanish for R_{-1}, and hence

$$|\operatorname{div} T_2| \le c_6|R - R_{-1}|\sigma_1(A) \le c_6(1-\lambda)\sigma_1(A)$$

by Lemma 2.1. Thus (15) gives

$$\mathcal{M}_3 \ge \sqrt{\lambda}\int_\Gamma \langle \nabla r, \nu\rangle \sigma_2(A) - c_6(1-\lambda)\mathcal{M}_1 \ge \big(3\lambda/2 - c_6(1-\lambda)\big)\mathcal{M}_1.$$

So we conclude that

$$\mathcal{G}(\Gamma) - |\mathbf{S}^5| \geq \left(\frac{15}{8} - c_6(1-\lambda)\right)|\Omega| - c_6(1-\lambda)\mathcal{M}_1.$$

For λ close to 1, the coefficient of $|\Omega|$ is positive; however, the negative term $c_6(1-\lambda)\mathcal{M}_1$ cannot in general be absorbed into the first term, since $\mathcal{M}_1(\Gamma)/|\Omega|$ is unbounded. Indeed, for geodesic balls $B_r \subset \mathbf{H}^n$, $\mathcal{M}_1(\partial B_r)/|B_r| \sim c_n/r^2 \to \infty$, as $r \to 0$. The same phenomenon occurs for thin tubes about a geodesic segment.

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