ISOPERIMETRIC AND TOTAL CURVATURE INEQUALITIES IN CARTAN-HADAMARD MANIFOLDS WITH NULLITY

MOHAMMAD GHOMI

Abstract. Using the Chern-Gauss-Bonnet theorem, we establish a sharp inequality for the total Gauss-Kronecker curvature of convex hypersurfaces in Cartan-Hadamard manifolds M^n with nullity index at least $n-3$. Consequently, the Euclidean isoperimetric inequality extends to M^n, which proves the Cartan-Hadamard conjecture for these spaces.

1. Introduction

A *Cartan-Hadamard manifold* M^n is a complete simply connected Riemannian space of nonpositive sectional curvature. These spaces are natural generalizations of Euclidean space $\mathbf{R}^n$. The *nullity index* $\mu(M)$, introduced by Chern and Kuiper [8], measures the flatness of M. More precisely, let

$$\mathcal{N}_p := \big\{Z \in T_pM \mid R(X,Y)Z = 0 \text{ for all } X, Y \in T_pM\big\}$$

be the nullity space of the curvature tensor at a point p. Then $\mu(M) := \inf_{p\in M} \dim(\mathcal{N}_p)$. So $M^n = \mathbf{R}^n$ if and only if $\mu(M) = n$. A convex hypersurface $\Gamma \subset M$ is the boundary of a compact convex set with nonempty interior. The *total curvature* of Γ, when it is $\mathcal{C}^{1,1}$, is defined as $\mathcal{G}(\Gamma) := \int_\Gamma GK$ where GK is the Gauss-Kronecker curvature. Let $|\mathbf{S}^{n-1}|$ denote the volume of the unit sphere in $\mathbf{R}^n$. We show

Theorem 1.1. *Let M^n be a Cartan-Hadamard manifold with $\mu(M) \geq n-3$. Then*

$$\mathcal{G}(\Gamma) \geq |\mathbf{S}^{n-1}|, \tag{1}$$

for any smooth ($\mathcal{C}^\infty$) convex hypersurface $\Gamma \subset M$.

In any Cartan-Hadamard manifold where the total curvature inequality (1) holds, the classical Euclidean isoperimetric inequality follows [15, Thm. 7.1]. Specifically, letting B^n denote the unit ball in $\mathbf{R}^n$, we obtain

Corollary 1.2. *For any bounded region $\Omega \subset M^n$ of finite perimeter,*

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\mathbf{S}^{n-1}|^n}{|B^n|^{n-1}}, \tag{2}$$

with equality only if Ω is isometric to a ball in $\mathbf{R}^n$.

Date: May 23, 2026 (Last Typeset).

2020 *Mathematics Subject Classification.* Primary 53C20, 53C21; Secondary 53C42, 52A40.

Key words and phrases. Cartan-Hadamard conjecture, convex hypersurfaces, manifolds with nullity, conullity, total Gauss-Kronecker curvature, Chern-Gauss-Bonnet theorem, isoperimetric inequality.

The author was supported by NSF grant DMS-2202337.

The *Cartan-Hadamard conjecture* [2, 6, 22] states that the isoperimetric inequality (2) holds in all Cartan-Hadamard manifolds. This has been established only in dimensions $n \leq 4$ [9, 23, 31], while the total curvature inequality (1) is known only for $n \leq 3$. Both inequalities hold in all dimensions when M^n has constant curvature or Γ is a geodesic sphere [4, 15]. Theorem 1.1 gives the first class of Cartan-Hadamard manifolds of dimension $n \geq 4$, beyond space forms, for which (1) holds, and also yields irreducible nonhomogeneous examples of dimension $n \geq 5$ where (2) holds. See [15, 24] for background and [16–18, 25] for more recent results.

The isoperimetric inequality is stable under Cartesian products [21, p. 335], [27, Sec. 3.3], but the total curvature inequality (1) is new even in the case of product manifolds, such as $\mathbf{H}^3 \times \mathbf{R}$, where $\mathbf{H}^3$ is hyperbolic 3-space. Manifolds with nullity arise naturally in connection with isometric immersions, totally geodesic foliations, and rigidity phenomena in Riemannian geometry [10, 20, 28]. There are many examples which do not split off a Euclidean factor, even locally. For conullity-two examples, see [3, 5, 30]; for related graph manifolds, whose universal covers give irreducible Cartan-Hadamard manifolds with nullity, see [11].

The proof of Theorem 1.1 is an application of the Chern-Gauss-Bonnet formula (6) together with Gauss' equation (8). When n is odd, we apply the formula to the even-dimensional hypersurface Γ, and when n is even, to the convex body C bounded by Γ. In either case, the nullity assumption implies that all terms in the Chern-Gauss-Bonnet integrands involving two or more ambient curvature factors vanish. Thus the integrands reduce to the Gauss-Kronecker curvature GK, up to a correction term which contains exactly one ambient sectional curvature factor and some principal curvatures of Γ. By the Cartan-Hadamard assumption, the ambient curvature factor is nonpositive, while convexity of Γ ensures that the principal curvatures are nonnegative. Hence the correction term is nonpositive, and we obtain (1).

2. The Chern-Gauss-Bonnet Theorem

We begin by recording the formulation of the Chern-Gauss-Bonnet theorem that we need. Let N^k be an oriented even-dimensional Riemannian manifold with metric $\langle \cdot, \cdot \rangle$, Levi-Civita connection ∇, and curvature tensors

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$R(X,Y,Z,W) := \langle R(X,Y)Z, W \rangle$. At each point $p \in N$, choose a positively oriented orthonormal basis $e_1, \ldots, e_k \in T_pN$. The corresponding curvature 2-forms of N are

$$\Omega_{ij}(X,Y) := R(X,Y,e_i,e_j).$$

The Pfaffian k-form associated to the skew-symmetric matrix $\Omega = (\Omega_{ij})$ is defined by

$$\Phi := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, \Omega_{\sigma(1)\sigma(2)} \wedge \cdots \wedge \Omega_{\sigma(k-1)\sigma(k)},$$

where S_k denotes the symmetric group on k elements, $\operatorname{sgn}(\sigma) = 1$ for even permutations, and $\operatorname{sgn}(\sigma) = -1$ for odd permutations. The associated normalized scalar function is

$$\mathrm{Pf}_N := c_k \Phi(e_1, \ldots, e_k), \tag{3}$$

where the constant $c_k := 1/k!$ is chosen so that $\mathrm{Pf}_{\mathbf{S}^k} \equiv 1$. Note that Pf_N is independent of the choice of the frame e_i, since Φ is invariant under conjugation by $\mathrm{SO}(k)$.

Now suppose that N is compact with boundary ∂N. Let $\nu = e_k$ be the outward unit normal of ∂N, and let $e_1, \dots, e_{k-1}$ be a positively oriented local orthonormal frame on ∂N. For $X \in T(\partial N)$, let $A(X) := \nabla_X \nu$ be the shape operator, and

$$\alpha_i(X) := \langle A(X), e_i \rangle$$

be the corresponding 1-forms on ∂N. So if e_i are the principal directions, which diagonalize A, then $\alpha_i(e_i) = \kappa_i$ are the corresponding principal curvatures of ∂N. The Chern boundary term is given by the transgression formula

$$\mathrm{TPf}_{\partial N} := \sum_{s=0}^{k/2-1} c_{k,s} \Phi_s(e_1, \dots, e_{k-1}), \tag{4}$$

where the $c_{k,s} := |\mathbf{S}^{k-1}|/((4\pi)^s\, |\mathbf{S}^{k-1-2s}|\, (k-1-2s)!)$ are normalizing constants, and Φ_s are $(k-1)$-forms given by combining the shape operator forms of ∂N and the ambient curvature forms of N as follows

$$\Phi_s := \sum_{\sigma \in S_{k-1}} \mathrm{sgn}(\sigma) \big(\alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(k-1-2s)}\big) \wedge \big(\Omega_{\sigma(k-2s)\sigma(k-2s+1)} \wedge \cdots \wedge \Omega_{\sigma(k-2)\sigma(k-1)}\big).$$

Here the second parenthesis is omitted when $s = 0$. Thus the first term in (4) reduces to the Gauss-Kronecker curvature:

$$c_{k,0}\Phi_0(e_1, \dots, e_{k-1}) = \det(A) = \kappa_1 \dots \kappa_{k-1} = GK. \tag{5}$$

Let χ denote the Euler characteristic. We have

Theorem 2.1 (Chern-Gauss-Bonnet). *Let N^k be a compact oriented even-dimensional Riemannian manifold, with possibly empty smooth boundary ∂N. Then*

$$\chi(N) = \frac{2}{|\mathbf{S}^k|} \int_N \mathrm{Pf}_N + \frac{1}{|\mathbf{S}^{k-1}|} \int_{\partial N} \mathrm{TPf}_{\partial N}, \tag{6}$$

where the boundary term is omitted when $\partial N = \emptyset$.

For proofs, see Chern [7], Allendoerfer-Weil [1], or Spivak [29, Addendum 2]. Our formulation here is consistent with Morgan-Johnson [26, Sec. 4.1] and Gilkey-Park [19, Thm. 1.3]. It quickly follows that

Corollary 2.2. *Let M^n be a Cartan-Hadamard manifold, and $\Gamma \subset M$ be a smooth convex hypersurface bounding a convex body C. If n is odd, then*

$$\int_\Gamma \mathrm{Pf}_\Gamma = |\mathbf{S}^{n-1}|.$$

If n is even, then

$$\int_\Gamma \mathrm{TPf}_\Gamma = |\mathbf{S}^{n-1}| - 2\frac{|\mathbf{S}^{n-1}|}{|\mathbf{S}^n|} \int_C \mathrm{Pf}_C\,.$$

Proof. Since M^n is Cartan-Hadamard, C is homeomorphic to the unit ball B^n, and consequently Γ is homeomorphic to $\mathbf{S}^{n-1}$. In particular, when n is odd, $\chi(\Gamma) = 2$ which yields the first equality above by Theorem 2.1 with $N = \Gamma$. The second equality also follows immediately from Theorem 2.1 with $N = C$, since $\chi(C) = 1$. □

3. Proof of Theorem 1.1

The nullity assumption has the following consequence for the curvature forms:

Lemma 3.1. *If $\mu(M) \geq n-3$, then for all indices $1 \leq i, j, k, \ell \leq n$,*

$$\Omega_{ij} \wedge \Omega_{k\ell} = 0, \tag{7}$$

with respect to any orthonormal basis.

Proof. Fix $p \in M$, and let $E_p := \mathcal{N}_p^{\perp}$. By hypothesis, $\dim(E_p) \leq 3$. Let $\pi \colon T_pM \to E_p$ be the orthogonal projection. Then

$$R(X, Y, Z, W) = R(\pi X, \pi Y, \pi Z, \pi W)$$

for all $X, Y, Z, W \in T_pM$. Indeed, if one of these vectors lies in $\mathcal{N}_p$, then R vanishes. This is evident from the definition of $\mathcal{N}_p$ when the vector appears in the third slot, and follows for the other slots by the symmetries of R. Consequently,

$$\Omega_{ij}(X, Y) = R(\pi X, \pi Y, \pi e_i, \pi e_j).$$

Thus, as a 2-form in X, Y, each Ω_{ij} is pulled back from the vector space E_p. It follows that $\Omega_{ij} \wedge \Omega_{k\ell}$ is pulled back from a 4-form on E_p. Since $\dim E_p \leq 3$, every 4-form on E_p vanishes, which completes the proof since p is arbitrary. □

We orient Γ by the outward unit normal ν, and use the convention $A(V) = \nabla_V \nu$ for the shape operator. Then the principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$ of Γ, and the Gauss-Kronecker curvature $GK = \kappa_1 \cdots \kappa_{n-1}$ are nonnegative. At each point $p \in \Gamma$, choose an orthonormal basis $e_1, \ldots, e_{n-1} \in T_p\Gamma$ which diagonalizes A. Let

$$K_{ij} := \Omega_{ij}(e_i, e_j)$$

be the sectional curvature of M with respect to the plane spanned by e_i, e_j. The Gauss equation gives $R^{\Gamma}_{ijij} = K_{ij} + \kappa_i \kappa_j$. Letting $\theta_1, \ldots, \theta_{n-1}$ be the coframe dual to $e_1, \ldots, e_{n-1}$, the Gauss equation may be written as

$$\Omega^{\Gamma}_{ij} = \Omega_{ij} + \kappa_i \kappa_j \, \theta_i \wedge \theta_j. \tag{8}$$

Substituting this equation in the expressions for the Pfaffians in the last section, we obtain the following result. Recall that C denotes the convex body bounded by Γ.

Proposition 3.2. *If n is odd, then*

$$\mathrm{Pf}_{\Gamma} \leq GK.$$

If n is even and $n \geq 4$, then

$$\mathrm{Pf}_C \equiv 0, \qquad \textit{and} \qquad \mathrm{TPf}_{\Gamma} \leq GK.$$

Proof. First suppose that n is odd. Then Γ has even dimension, and Pf_Γ is a linear combination of wedge products of the forms Ω^Γ_{ij} by (3). After substituting the Gauss equation (8), each term contains some number of ambient curvature forms Ω_{ij} and some number of factors $\kappa_i\kappa_j\,\theta_i\wedge\theta_j$. By Lemma 3.1, all terms with two or more ambient curvature forms vanish. Thus only the terms with no ambient curvature form or exactly one ambient curvature form can contribute.

The contribution to Pf_Γ from terms with no ambient curvature form is precisely GK. Now consider a term with exactly one ambient curvature form, say Ω_{ij}. Since all the remaining factors come from the diagonal second fundamental form, they contribute multiples of $\theta_a\wedge\theta_b$ and supply all coframe directions except θ_i,θ_j. Consequently, when the resulting $(n-1)$-form is evaluated on $e_1,\dots,e_{n-1}$, we obtain

$$\Omega_{ij}(e_i,e_j)\prod_{\ell\neq i,j}\kappa_\ell = K_{ij}\prod_{\ell\neq i,j}\kappa_\ell \leq 0,$$

where the last inequality holds since $K_{ij}\leq 0$ by the Cartan-Hadamard assumption, and $\kappa_\ell\geq 0$ by convexity of Γ. Hence, for some constant $c_n>0$,

$$\mathrm{Pf}_\Gamma = GK + c_n\sum_{1\leq i<j\leq n-1}K_{ij}\prod_{\ell\neq i,j}\kappa_\ell \leq GK. \tag{9}$$

Next suppose that n is even and $n\geq 4$. By (3), Pf_C consists of wedge products of $n/2$ ambient curvature forms. Since $n/2\geq 2$, each such term contains at least two ambient curvature forms, and hence vanishes by Lemma 3.1. So $\mathrm{Pf}_C\equiv 0$.

Finally, to evaluate TPf_Γ, note that in (4), the factors α_i coming from the second fundamental form are the 1-forms $\kappa_i\theta_i$. The first term, which contains no ambient curvature form, contributes the Gauss-Kronecker curvature GK by (5). As above, Lemma 3.1 eliminates all terms with two or more ambient curvature forms. If a term contains exactly one ambient curvature form Ω_{ij}, then again the remaining second fundamental form factors supply all coframe directions except θ_i,θ_j. Hence only the component $\Omega_{ij}(e_i,e_j)=K_{ij}$ contributes. Thus, for a constant $c'_n>0$,

$$\mathrm{TPf}_\Gamma = GK + c'_n\sum_{1\leq i<j\leq n-1}K_{ij}\prod_{\ell\neq i,j}\kappa_\ell \leq GK, \tag{10}$$

which completes the proof. □

Now if n is odd, then Corollary 2.2 together with Proposition 3.2 gives

$$|\mathbf{S}^{n-1}| = \int_\Gamma \mathrm{Pf}_\Gamma \leq \int_\Gamma GK = \mathcal{G}(\Gamma). \tag{11}$$

If $n=2$, then the ordinary Gauss-Bonnet theorem gives $2\pi=\int_C K+\int_\Gamma GK$, and since $K\leq 0$, it follows that $\mathcal{G}(\Gamma)=\int_\Gamma GK\geq 2\pi=|\mathbf{S}^1|$. Finally suppose that $n\geq 4$ is even. Then Corollary 2.2 and Proposition 3.2 give

$$|\mathbf{S}^{n-1}| = \int_\Gamma \mathrm{TPf}_\Gamma \leq \int_\Gamma GK = \mathcal{G}(\Gamma), \tag{12}$$

which completes the proof.

4. Notes

Note 4.1. The nullity assumption $\mu(M) \geq n-3$ was used in the proof of Theorem 1.1 only to derive (7), which is a formally weaker condition at each point. Thus the total curvature inequality (1) holds whenever (7) holds on M. More specifically, it is enough that (7) holds on Γ when n is odd, and on the convex body C when n is even.

Note 4.2. If equality holds in the total curvature inequality (1), for $n \geq 3$, then equality holds in the corresponding estimate (11) or (12), according as n is odd or even. Hence equality also holds in the corresponding pointwise inequality (9) or (10), which yields

$$K_{ij} \prod_{\ell \neq i,j} \kappa_\ell = 0$$

for every $i < j$. In particular, when Γ is strictly convex, i.e., $\kappa_\ell > 0$ for all ℓ, equality in (1) forces the sectional curvatures of M to vanish on tangent planes of Γ, which in turn yields that C is flat [12].

Note 4.3. The total curvature of an arbitrary convex hypersurface $\Gamma \subset M^n$ may be defined as the limit of the total curvatures of its outer parallel hypersurfaces, which are $\mathcal{C}^{1,1}$ [15]. With this definition, the total curvature functional $\mathcal{G}$ is continuous on the space of convex hypersurfaces in Cartan-Hadamard manifolds, with respect to Hausdorff distance [13]. Since smooth convex hypersurfaces are dense in this space [14, Lem. 3.9], Theorem 1.1 extends to all convex hypersurfaces, with no smoothness assumption.

Acknowledgment

The author thanks John Stavroulakis for useful discussions.

References

[1] C. B. Allendoerfer and A. Weil, *The Gauss-Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc. **53** (1943), 101–129. MR7627 ↑3

[2] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), no. 5, Aii, A279–A281. MR0407905 ↑2

[3] E. Boeckx, O. Kowalski, and L. Vanhecke, *Riemannian manifolds of conullity two*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996. MR1462887 ↑2

[4] A. Borbély, *On the total curvature of convex hypersurfaces in hyperbolic spaces*, Proc. Amer. Math. Soc. **130** (2002), no. 3, 849–854. MR1866041 ↑2

[5] T. G. Brooks, *3-manifolds with constant Ricci eigenvalues* $(\lambda, \lambda, 0)$, Geom. Dedicata **219** (2025), no. 1, Paper No. 13, 25. MR4842957 ↑2

[6] Yu. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 285, Springer-Verlag, Berlin, 1988. Translated from the 1980 Russian original by A. B. Sosinskiĭ, Springer Series in Soviet Mathematics. MR936419 ↑2

[7] S.-S. Chern, *On the curvatura integra in a Riemannian manifold*, Ann. of Math. (2) **46** (1945), 674–684. MR14760 ↑3

[8] S.-S. Chern and N. H. Kuiper, *Some theorems on the isometric imbedding of compact Riemann manifolds in euclidean space*, Ann. of Math. (2) **56** (1952), 422–430. MR50962 ↑1

[9] C. B. Croke, *A sharp four-dimensional isoperimetric inequality*, Comment. Math. Helv. **59** (1984), no. 2, 187–192. MR749103 ↑2

[10] D. Ferus, *On the completeness of nullity foliations*, Michigan Math. J. **18** (1971), 61–64. MR279733 ↑2

[11] L. A. Florit and W. Ziller, *Manifolds with conullity at most two as graph manifolds*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 5, 1313–1333. MR4174849 ↑2

[12] M. Ghomi, *Convexity and rigidity of hypersurfaces in Cartan-Hadamard manifolds*, Asian J. Math. **29** (2025), no. 6, 761–772. ↑6

[13] ———, *Continuity of total curvatures of riemannian hypersurfaces*, Proc. Amer. Math. Soc., to appear (2026). `https://arxiv.org/abs/2511.15955`. ↑6

[14] M. Ghomi, J. A. Hoisington, M. Raffaelli, and J. I. Stavroulakis, *Total absolute curvature and rigidity of surfaces in Cartan-Hadamard manifolds*, arXiv:2604.25024 (2026). `https://arxiv.org/abs/2604.25024`. ↑6

[15] M. Ghomi and J. Spruck, *Total curvature and the isoperimetric inequality in Cartan-Hadamard manifolds*, J. Geom. Anal. **32** (2022), no. 2, Paper No. 50, 54pp. MR4358702 ↑1, 2, 6

[16] ———, *Total mean curvatures of Riemannian hypersurfaces*, Adv. Nonlinear Stud. **23** (2023), no. 1, Paper No. 20220029, 10. MR4530499 ↑2

[17] M. Ghomi and J. I. Stavroulakis, *A local isoperimetric inequality for balls with nonpositive curvature*, Int. Math. Res. Not. IMRN **6** (2026), rnag043. ↑2

[18] ———, *Total curvature of convex hypersurfaces in Cartan-Hadamard manifolds*, arXiv:2601.13280 (2026). `https://arxiv.org/abs/2601.13280`. ↑2

[19] P. Gilkey and J. H. Park, *Analytic continuation, the Chern-Gauss-Bonnet theorem, and the Euler-Lagrange equations in Lovelock theory for indefinite signature metrics*, J. Geom. Phys. **88** (2015), 88–93. MR3293398 ↑3

[20] C. Gorodski and F. Guimarães, *The κ-nullity of Riemannian manifolds and their splitting tensors*, Ann. Mat. Pura Appl. (4) **202** (2023), no. 6, 2561–2583. MR4648538 ↑2

[21] M. Gromov, *Sign and geometric meaning of curvature*, Rend. Sem. Mat. Fis. Milano **61** (1991), 9–123 (1994). MR1297501 ↑2

[22] ———, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser Boston Inc., Boston, MA, 1999. MR2000d:53065 ↑2

[23] B. Kleiner, *An isoperimetric comparison theorem*, Invent. Math. **108** (1992), no. 1, 37–47. MR1156385 ↑2

[24] B. R. Kloeckner and G. Kuperberg, *The Cartan-Hadamard conjecture and the Little Prince*, Rev. Mat. Iberoam. **35** (2019), no. 4, 1195–1258. MR3988083 ↑2

[25] J. Li, Z. Lin, and L. Xu, *A total curvature estimate of closed hypersurfaces in non-positively curved symmetric spaces*, arXiv:2501.17146 (2025). `https://arxiv.org/abs/2501.17146`. ↑2

[26] F. Morgan and D. L. Johnson, *Some sharp isoperimetric theorems for Riemannian manifolds*, Indiana Univ. Math. J. **49** (2000), no. 3, 1017–1041. MR1803220 ↑3

[27] A. Ros, *The isoperimetric problem*, Global theory of minimal surfaces, 2005, pp. 175–209. MR2167260 ↑2

[28] A. Rosenthal, *Riemannian manifolds of constant k-nullity*, Proc. Amer. Math. Soc. **22** (1969), 473–475. MR254784 ↑2

[29] M. Spivak, *A comprehensive introduction to differential geometry. Vol. V*, Second, Publish or Perish Inc., Wilmington, Del., 1979. MR532834 (82g:53003e) ↑3

[30] J. Van Hook, *On the geometry of conullity two manifolds*, Differential Geom. Appl. **92** (2024), Paper No. 102081, 21. MR4669463 ↑2

[31] A. Weil, *Sur les surfaces a courbure negative*, CR Acad. Sci. Paris **182** (1926), no. 2, 1069–71. ↑2

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332
Email address: `ghomi@math.gatech.edu`
URL: `www.math.gatech.edu/~ghomi`