A COUNTEREXAMPLE TO NIRENBERG'S CONJECTURE ON CLOSED ASYMPTOTIC CURVES

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Abstract. We construct a smooth embedded annulus in Euclidean 3-space with negative Gauss curvature in its interior whose boundary consists of strictly convex curves in planes tangent to the annulus, and which contains closed asymptotic curves. This gives a counterexample to a 1962 conjecture of Nirenberg. The construction is based on Lelieuvre's formula, which yields a closed curve that is then realized as an asymptotic curve on a ruled annulus. The second asymptotic curve is obtained from a periodic solution of a Riccati equation.

1. Introduction

In 1938, Alexandrov [1] showed that real analytic tight surfaces in Euclidean space $\mathbf{R}^3$ are isometrically rigid. In his celebrated program [7–9, 16] to extend Alexandrov's theorem to smooth surfaces, Nirenberg [13] conjectured that a negatively curved annulus bounded by convex curves in tangent planes cannot contain a closed asymptotic curve. Here we construct a counterexample.

Theorem 1.1. *There exists a smooth embedded annulus $M \subset \mathbf{R}^3$ whose interior has negative Gauss curvature and contains two distinct closed asymptotic curves. The boundary components of M are strictly convex curves in parallel planes, each tangent to M along the corresponding boundary component.*

The construction uses Lelieuvre's classical formula [11] to produce a closed curve and then realizes it as an asymptotic curve on a ruled annulus. Modifying the height near its ends produces the required boundary tangency while preserving negative curvature. The asymptotic curves transverse to the rulings satisfy a Riccati equation. Using the first curve as a known solution, we reduce this equation to a linear equation and obtain a second closed curve. The construction is explicit, and the required estimates are proved analytically. The accompanying Mathematica notebook [3] checks the computations and generates the figures below.

The significance of asymptotic curves in Nirenberg's program stems from their role as characteristics of the hyperbolic PDE governing isometric immersions in the negatively curved region. Nirenberg established rigidity under additional assumptions, including

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the requirement that each component of this region contain at most one closed asymptotic curve, and his conjecture sought to remove this requirement. Motivated by this problem, Kovaleva [10] constructed a negatively curved strip containing a closed asymptotic curve, while Panov [15] and Arnold [2] investigated restrictions on closed asymptotic curves on negatively curved graph surfaces. Ghomi and Raffaelli [6] related their topology to the Gauss map, and Ni, Zhang, and Zhang [12] resolved similar conjectures of Pogorelov and Toponogov for complete saddle surfaces. Ding [5] answered a question posed in [6] on a weaker form of Nirenberg's conjecture.

2. Construction of the Annulus

2.1. **Lelieuvre's formula.** By *smooth* we mean $\mathcal{C}^\infty$. A smooth curve $\Gamma\colon (a,b) \to \mathbf{R}^3$ is *regular* if its derivative $\Gamma' \neq 0$. A regular curve Γ on a smooth surface $S \subset \mathbf{R}^3$ is *asymptotic* if its normal curvature $\mathrm{II}(\Gamma', \Gamma') = 0$, or equivalently

$$\langle \Gamma'', N \rangle = 0,$$

where N is a unit normal vector field of S. A parametrization $X(u,v)$ of S is *asymptotic* if $\langle X_{uu}, N\rangle = \langle X_{vv}, N\rangle = 0$. Lelieuvre's formula [4, Sec. 1.3; 11, p. 126; 14] gives a convenient description for such parametrizations when the Gauss curvature $K < 0$: after interchanging u and v if necessary, the normal field $\mu := (-K)^{-1/4} N$ satisfies $X_u = \mu \times \mu_u$, $X_v = \mu_v \times \mu$. In particular, if $v = v_0$ is fixed and $\Gamma(u) := X(u, v_0)$, $\mu(u) := \mu(u, v_0)$, we may write

$$\Gamma' = \mu \times \mu'. \tag{1}$$

We will use this formula in the reverse direction: start with a smooth closed curve $\mu\colon \mathbf{R}/(2\pi\mathbf{Z}) \to \mathbf{R}^3$ satisfying $\mu \times \mu' \neq 0$, define Γ by (1), and construct a smooth surface through Γ which is orthogonal to μ. On such a surface, Γ is asymptotic, since $\langle \Gamma'', \mu\rangle = \langle \mu \times \mu'', \mu\rangle = 0$. More specifically, we choose a transverse vector field ξ along Γ orthogonal to μ. Then

$$\Gamma(t) + r\xi(t) \tag{2}$$

is a ruled surface near $r = 0$. Along Γ its tangent vectors are Γ' and ξ, both perpendicular to μ. Hence μ is normal to this surface along Γ, and Γ is asymptotic on it. We need to choose μ and ξ so that Γ closes.

2.2. **The first asymptotic curve.** To carry out the construction described above, identify $\mathbf{R}^3$ with $\mathbf{C} \times \mathbf{R}$. We choose the ruling direction

$$\xi(t) := (-e^{\mathrm{i}t}, 1).$$

The vertical component makes the height monotone along each ruling, while the rotation of the horizontal component will render distant horizontal sections of the ruled surface nearly circular, which aids embeddedness and the convexity of the ends. A vector field with positive vertical component is perpendicular to ξ precisely when it is a positive multiple of $\big(e^{\mathrm{i}t}(1 - \mathrm{i}A(t)), 1\big)$ for some real function A. So we set

$$\mu(t) := \sqrt{B(t)}\,\big(e^{\mathrm{i}t}(1 - \mathrm{i}A(t)), 1\big),$$

where $B > 0$ will be chosen below. Thus A determines the direction of μ, while B only rescales $\mu \times \mu'$ pointwise, giving us additional freedom to control Γ. We seek A and B in the form

$$A(t) \coloneqq 1 + a\sin(nt), \qquad B(t) \coloneqq 1 + b\cos(nt),$$

where $n \geq 2$ is an integer and $0 < a, b < 1$. Define our first asymptotic curve Γ_1 by $\Gamma_1' \coloneqq \mu \times \mu'$ and write

$$\Gamma_1(t) =: \big(F(t), G(t)\big) \in \mathbf{C} \times \mathbf{R}.$$

A direct computation gives

$$F' = Be^{\mathrm{i}t}(A' - 1 + \mathrm{i}A), \qquad G' = B(1 + A^2 - A'). \tag{3}$$

Note that $\Gamma_1(0) = \Gamma_1(2\pi)$ precisely when $\int_0^{2\pi} F' = \int_0^{2\pi} G' = 0$. By definition, F' is a sum of terms $c\,e^{\mathrm{i}kt}$ for $k = 1,\ 1 \pm n,\ 1 \pm 2n$. Since $n \geq 2$, these values of k are nonzero, and so $\int_0^{2\pi} F' = 0$. On the other hand, $\frac{1}{2\pi}\int_0^{2\pi} G'(t)\,dt = 2 + a^2/2 - nab/2$, which vanishes precisely when $b = (4 + a^2)/(na)$. We take

$$n = 12, \qquad a = \frac{2}{5}, \qquad b = \frac{13}{15}. \tag{4}$$

Since $B > 0$ forces $b < 1$, the closing condition requires $na > 4 + a^2$; for $a = 2/5$ the smallest admissible value is $n = 11$. Taking $n = 12$ instead yields wider margins in the estimates below. Thus

$$A(t) = 1 + \frac{2}{5}\sin(12t), \qquad B(t) = 1 + \frac{13}{15}\cos(12t). \tag{5}$$

Fixing the constants of integration in the antiderivatives of (3) so that $\int_0^{2\pi} F = \int_0^{2\pi} G = 0$, we obtain

$$\begin{aligned} F(t) &= \left(1 - \frac{27}{25}\mathrm{i}\right)e^{\mathrm{i}t} + \frac{-13 + 53\mathrm{i}}{330}e^{-11\mathrm{i}t} + \frac{1 - 5\mathrm{i}}{30}e^{13\mathrm{i}t} + \frac{143\mathrm{i}}{3450}e^{-23\mathrm{i}t} - \frac{169\mathrm{i}}{3750}e^{25\mathrm{i}t}, \\ G(t) &= -\frac{379}{1500}\sin 12t - \frac{1}{15}\cos 12t - \frac{9}{100}\sin 24t - \frac{13}{900}\cos 24t - \frac{13}{13500}\sin 36t. \end{aligned} \tag{6}$$

Hence Γ_1 is closed. It is also regular because $|F'| = B\sqrt{(A' - 1)^2 + A^2} \geq BA > 0$.

2.3. **The ruled surface.** Next we construct the ruled surface (2) along Γ_1. The height of $\Gamma_1(t) + r\xi(t)$ is $G(t) + r$, which will be convenient to use as the parameter along the rulings. So we set $s \coloneqq G(t) + r$ and define

$$X_0(t, s) \coloneqq \Gamma_1(t) + \big(s - G(t)\big)\xi(t) = \big(F(t) + (G(t) - s)e^{\mathrm{i}t}, s\big).$$

Along the curve $\Gamma_1(t) = X_0(t, G(t))$, μ is perpendicular to $\Gamma_1' = \mu \times \mu'$, and to the ruling direction ξ by construction. Furthermore, if $\Gamma_1' = (F', G')$ were a multiple of $\xi = (-e^{\mathrm{i}t}, 1)$, then $F' + G'e^{\mathrm{i}t}$ would vanish; but (3) gives $F' + G'e^{\mathrm{i}t} = BAe^{\mathrm{i}t}(A + \mathrm{i}) \neq 0$. Thus Γ_1' and ξ are linearly independent, and so μ is a normal field of X_0 along Γ_1. It follows from (1) that Γ_1 is an asymptotic curve of X_0.

2.4. **Modification near the boundary.** Here we restrict the ruled surface X_0 to $|s| \le 200$, and modify its vertical coordinate near $s = \pm 200$ so that the resulting boundary curves lie in horizontal tangent planes. The specific height function z below is chosen so that this flattening also preserves the negative curvature of the interior, via the inequality $sz''(s) \le 0$ (as shown in the proof of Proposition 3.3). Let

$$P(v) := \begin{cases} \dfrac{v^2}{75} e^{\,1-25/v}, & v > 0, \\ 0, & v \le 0, \end{cases}$$

and set $z(s) := s - \operatorname{sign}(s) P(|s| - 175)$, $-200 \le s \le 200$. Here $\operatorname{sign}(0) = 0$. The function P is smooth and all its derivatives vanish at zero. Moreover, for $0 < v \le 25$,

$$P'(v) = \frac{2v + 25}{75} e^{\,1-25/v}, \qquad P''(v) = \frac{e^{\,1-25/v}}{75} \left(2 + \frac{50v + 625}{v^2}\right) > 0.$$

Consequently $0 < P'(v) < 1$ for $0 < v < 25$, and $P'(25) = 1$. Hence $z' > 0$ on $(-200, 200)$, $z'(\pm 200) = 0$, and $sz''(s) \le 0$. Also $z(s) = s$ for $|s| \le 175$. We set

$$X(t, s) := \Big(F(t) + (G(t) - s)e^{\mathrm{i}t}, z(s)\Big),$$

for $(t, s) \in \mathbf{R}/(2\pi\mathbf{Z}) \times [-200, 200]$. It agrees with X_0 on $|s| \le 175$, so the curve Γ_1 and the ruled geometry near it are unchanged. Finally we define our annulus by

$$M := X\big(\mathbf{R}/(2\pi\mathbf{Z}) \times [-200, 200]\big).$$

3. Properties of the Annulus

3.1. **Embeddedness and smoothness.** Write $w(t, s) := F(t) + (G(t) - s)e^{\mathrm{i}t}$ for the horizontal component of X at height $z(s)$. Set $p := BA^2$, $q := BA + G$. Then (3) gives

$$w_t = e^{\mathrm{i}t}\big(p + \mathrm{i}(q - s)\big), \qquad w_s = -e^{\mathrm{i}t}. \tag{7}$$

To control the chords of the horizontal sections of X, for t_1, $t_2 \in \mathbf{R}$ we set $m := (t_1 + t_2)/2$, $\alpha := (t_1 - t_2)/2$, and define

$$\mathcal{E}(m, \alpha) := \frac{1}{2 \sin \alpha} \int_{-\alpha}^{\alpha} \Big(p(m + \rho) \cos \rho - \big(q(m + \rho) - 1\big) \sin \rho\Big)\, d\rho. \tag{8}$$

Since $\mathcal{E}(m, \alpha) \to p(m)$ as $\alpha \to 0$, we set $\mathcal{E}(m, 0) := p(m)$. Integrating (7) yields $\operatorname{Re}\big(e^{-\mathrm{i}m}(w(t_1, s) - w(t_2, s))\big) = 2 \sin \alpha\, \mathcal{E}(m, \alpha)$, where the term involving $1 - s$ drops out because $\sin \rho$ is odd. So the horizontal sections of X are simple closed curves wherever $\mathcal{E} > 0$. We will use the Fourier expansions

$$p = \frac{27}{25} + \frac{338}{375} \cos 12t + \frac{4}{5} \sin 12t - \frac{2}{25} \cos 24t + \frac{26}{75} \sin 24t - \frac{13}{375} \cos 36t, \tag{9}$$

$$q = 1 + \frac{4}{5} \cos 12t + \frac{221}{1500} \sin 12t - \frac{13}{900} \cos 24t + \frac{1}{12} \sin 24t - \frac{13}{13500} \sin 36t. \tag{10}$$

Lemma 3.1. *For the parameters* (4), $\mathcal{E}(m, \alpha) > 0$ *for every* $m \in \mathbf{R}$ *and* $0 \le \alpha \le \pi/2$.

Proof. We first observe that $p > 2/25$ and $|q-1| < 21/20$. Indeed, if $\sin 12t \geq -1/2$, then (5) gives $p \geq (4/5)^2(2/15) > 2/25$. Otherwise $|\cos 12t| < \sqrt{3}/2 < 7/8$, so $p > (3/5)^2(29/120) > 2/25$. Summing the absolute values of the coefficients in (10) gives $|q-1| < 21/20$. For $0 < \alpha \leq 1/8$, (8) therefore yields

$$\mathcal{E}(m,\alpha) > \frac{2}{25} - \frac{21}{20}\frac{1-\cos\alpha}{\sin\alpha} = \frac{2}{25} - \frac{21}{20}\tan\frac{\alpha}{2} \geq \frac{1}{100}.$$

Here $\tan(\alpha/2) \leq 1/15$ follows from $\sin x \leq x$ and $\cos x \geq 1 - x$ for $0 \leq x \leq 1/16$. Now suppose $1/8 \leq \alpha \leq \pi/2$. For $k = 12, 24, 36$, define

$$C_k \coloneqq \frac{k\cot\alpha\sin(k\alpha) - \cos(k\alpha)}{k^2-1}, \qquad S_k \coloneqq \frac{\cot\alpha\sin(k\alpha) - k\cos(k\alpha)}{k^2-1}.$$

Writing $p_j^c, p_j^s, q_j^c, q_j^s$ for the cosine and sine coefficients of the $12j$ harmonics in (9) and (10), direct integration gives

$$\mathcal{E} = \frac{27}{25} + \sum_{j=1}^{3}\Big(\big(p_j^c C_{12j} - q_j^s S_{12j}\big)\cos(12jm) + \big(p_j^s C_{12j} + q_j^c S_{12j}\big)\sin(12jm)\Big).$$

The coefficients satisfy the elementary bounds

j	1	2	3
$\sqrt{(p_j^c)^2 + (p_j^s)^2}$	$< 121/100$	$< 9/25$	$< 7/200$
$\sqrt{(q_j^c)^2 + (q_j^s)^2}$	$< 41/50$	$< 17/200$	$< 1/1000$

Since $0 \leq \cot\alpha \leq 8$, we have $|C_k| \leq (8k+1)/(k^2-1)$ and $|S_k| \leq (8+k)/(k^2-1)$. Applying Cauchy–Schwarz separately to the terms multiplying C_{12j} and S_{12j} consequently gives

$$\mathcal{E} \geq \frac{27}{25} - \frac{1029}{1100} - \frac{361}{2875} - \frac{10159}{1295000} > \frac{1}{100}.$$

Finally, $\mathcal{E}(m,0) = p(m) > 2/25$. □

Proposition 3.2. *X is a smooth embedding.*

Proof. Since z is strictly increasing, points with different values of s have different heights. So to establish that X is injective it suffices to check that each horizontal section $t \mapsto w(t,s)$ is simple. Distinct points of $\mathbf{R}/2\pi\mathbf{Z}$ have representatives in $\mathbf{R}$ with $0 < t_1 - t_2 \leq \pi$, or $\alpha = (t_1-t_2)/2 \in (0, \pi/2]$. Thus $w(t_1,s) \neq w(t_2,s)$ by Lemma 3.1, as desired. The sections are also regular, since $|w_t| \geq p = BA^2 > 0$ by (7). It remains to check that X is an immersion. By (7), $X_t = \big(e^{\mathrm{i}t}(p + \mathrm{i}(q-s)), 0\big)$, which never vanishes since $p > 0$, while $X_s = (-e^{\mathrm{i}t}, z')$. So these vectors are linearly independent in the interior, where $z' > 0$. At $s = \pm 200$, where $z' = 0$, they are parallel only if $q - s = 0$, while (10) gives $|q| < 3$. Thus X is indeed an immersion up to the boundary. □

3.2. **Curvature and boundary geometry.** Next we compute the Gauss curvature of M. By (7), $X_t \times X_s = \big(z' e^{\mathrm{i}t}(q - s - \mathrm{i}p), q - s\big)$. For horizontal vectors, let det denote the oriented planar determinant, and put

$$D \coloneqq \det(w_t, w_{tt}) = p^2 + (q-s)^2 + pq' - (q-s)p'. \tag{11}$$

Thus D is the numerator of the signed curvature of a horizontal section. The coefficients of the second fundamental form relative to $X_t \times X_s$ are

$$\langle X_{tt}, X_t \times X_s \rangle = -z'D, \quad \langle X_{ts}, X_t \times X_s \rangle = z'p, \quad \langle X_{ss}, X_t \times X_s \rangle = z''(q-s). \tag{12}$$

Consequently the Gauss curvature is

$$K = \frac{-(z')^2p^2 - z'z''(q-s)D}{\Big((z')^2\big(p^2+(q-s)^2\big) + (q-s)^2\Big)^2}. \tag{13}$$

Proposition 3.3. *The interior of M has negative Gauss curvature. Its boundary components are strictly convex curves in two horizontal planes, and these planes are tangent to M along the boundary.*

Proof. In the region $|s| \le 175$, we have $z' = 1$ and $z'' = 0$. Thus (13) reduces to $K = -p^2/\big(p^2+2(q-s)^2\big)^2 < 0$. To treat the end regions, we use the bounds $0 < p < 4, |q| < 3, |p'| < 46, |q'| < 21$. Indeed, (5) gives $3/5 \le A \le 7/5$, $2/15 \le B \le 28/15$, $|A'| \le 24/5$, and $|B'| \le 52/5$. Together with

$$p' = B'A^2 + 2BAA', \qquad q' = B'A + B(1+A^2),$$

these inequalities yield $|p'| \le 5684/125 < 46$ and $|q'| \le 7532/375 < 21$. The bound for q follows directly from (10). For $175 \le |s| \le 200$, we have $|q-s| > 172$. Therefore

$$D \ge (q-s)^2 - |q-s|\,|p'| - p|q'| > 172(172-46) - 84 = 21588 > 0. \tag{14}$$

Furthermore, $q-s$ has sign opposite to s. Since $sz''(s) \le 0$, it follows that $z''(q-s) \ge 0$. In the interior $z' > 0$, and hence the numerator of (13) is strictly negative. So $K < 0$ in the interior. At $s = \pm 200$, both tangent vectors are horizontal and linearly independent, as was checked in Proposition 3.2. Thus the boundary tangent planes are the horizontal planes through $z(\pm 200)$. The boundary curves are simple by Proposition 3.2 and have positive signed curvature by (14). Hence they are strictly convex. □

4. Proof of Theorem 1.1

By Propositions 3.2 and 3.3, the surface M we constructed in Section 2 is a smooth embedded annulus with negative curvature in its interior, whose boundary components are strictly convex curves in parallel tangent planes. It remains only to check that M contains two distinct closed asymptotic curves. The first curve is Γ_1, constructed in Section 2.2. Indeed, by (6) we have $|G| < 1/2$, so the graph $s = G$ lies in the region $|s| < 175$ of M, where $X = X_0$. Thus

$$\Gamma_1(t) = X(t, G(t))$$

is asymptotic by Section 2.3. Its simplicity and smooth closure follow from Proposition 3.2 and the periodicity of F and G. To construct the second curve, Γ_2, we record the following elementary bounds, which are immediate from (5):

$$\frac{3}{5} \le A \le \frac{7}{5}, \qquad \frac{2}{15} \le B \le \frac{28}{15}. \tag{15}$$

For $|s| < 175$, a regular curve $(t(\tau), s(\tau))$ in the parameter space of X is asymptotic exactly when $\dot{t}\,(-D\,\dot{t} + 2p\,\dot{s}) = 0$, by (12), where dots denote derivatives with respect

to τ. Thus the asymptotic curves form two families: the rulings, $t =$ constant, and the graphs $s = s(t)$ satisfying the Riccati equation

$$s' = \frac{D}{2p} = \frac{s^2 + (p' - 2q)\, s + p^2 + q^2 + pq' - qp'}{2p}, \tag{16}$$

where we have expanded D via (11).

Lemma 4.1. *A graph $s = G + h$ contained in the region $|s| < 175$ of M, where h is nowhere vanishing, is asymptotic if and only if $f := A\sqrt{B}/h$ satisfies the linear equation*

$$f' = \frac{f}{A} - \frac{1}{2A\sqrt{B}}. \tag{17}$$

Proof. The graph $s = G$ is asymptotic, so $D(t, G) = 2pG'$. Using $q - G = BA$ in (11), we find $D(t, G + h) = D(t, G) + (p' - 2BA)h + h^2$. By substitution into (16), the graph $s = G + h$ is asymptotic if and only if

$$h' = \left(\frac{p'}{2p} - \frac{1}{A}\right) h + \frac{h^2}{2p}.$$

Since $p = BA^2$, setting $h = A\sqrt{B}/f$ transforms this equation into (17). □

We will use the periodic solution of (17) to construct the second curve. This solution is given by

$$f(t) := \int_t^\infty \frac{e^{-\int_t^\sigma d\rho/A(\rho)}}{2A(\sigma)\sqrt{B(\sigma)}}\, d\sigma, \qquad h(t) := \frac{A(t)\sqrt{B(t)}}{f(t)}, \tag{18}$$

where the integral converges since A is bounded above.

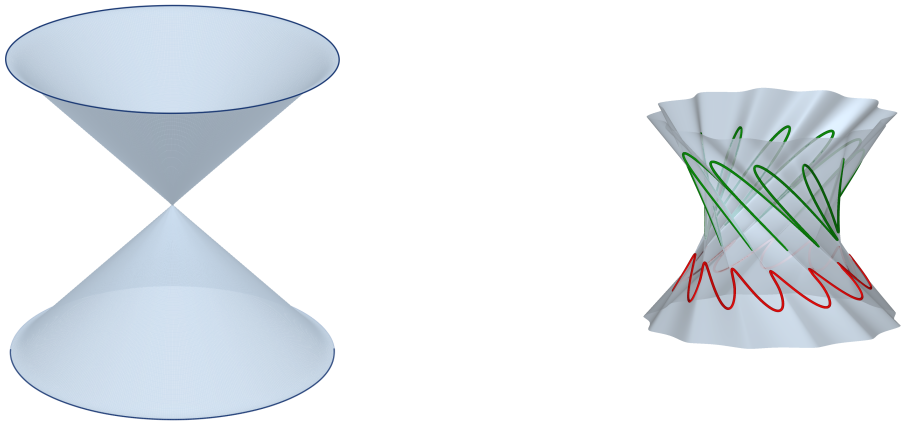
Proposition 4.2. *The curve*

$$\Gamma_2(t) := X\big(t, G(t) + h(t)\big) \tag{19}$$

is a simple closed asymptotic curve of M disjoint from Γ_1.

Proof. The periodicity of A and B shows that f, and hence h, is $\pi/6$-periodic, and differentiating (18) shows that f satisfies (17). At a minimum or maximum of f, (17) gives $f = 1/(2\sqrt{B})$. Hence (15) yields $1/(2\sqrt{28/15}) \le f \le 1/(2\sqrt{2/15})$. In particular $f > 0$, so $h > 0$, and the same bounds give $h \le 392/75$. Together with $|G| < 1/2$, it follows that the graph $s = G + h$ lies in the region $|s| < 175$ of M. So, since f is positive and satisfies (17), Lemma 4.1 shows that Γ_2 is asymptotic. The linear equation (17) has analytic coefficients, so f and h are real analytic, and the periodicity of F, G, and h gives smooth closure of Γ_2. Its parameter graph is regular and embedded, and is disjoint from the graph $s = G$ because $h > 0$. Proposition 3.2 transfers these properties to the images in M. □

Due to the choice of the frequency $n = 12$ in (4), M, Γ_1, and Γ_2 have 12-fold rotational symmetry. Figure 1(a) shows M. The ruled waist, which contains Γ_1 and Γ_2, is too small to appear in Figure 1(a) and is shown in Figure 1(b). Figure 2 confirms that Γ_1 and Γ_2 are images of smooth curves in the parameter space, and so are indeed smooth.



(a) (b)

FIGURE 1. (a) The annulus M. (b) Close-up of the ruled waist containing the asymptotic curves, Γ_1 (red) and Γ_2 (green).

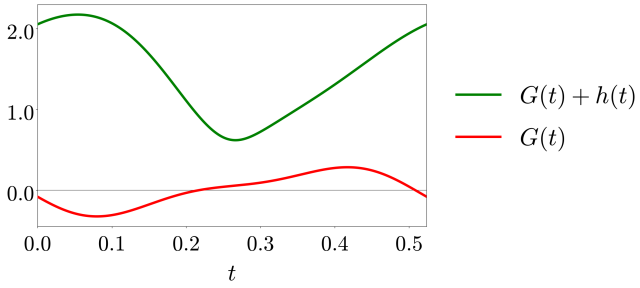


FIGURE 2. Heights of the asymptotic curves Γ_1 and Γ_2 in the parameter space of M, over one of the 12 symmetry sectors.

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This paper is the culmination of six months of work, partly carried out with AI tools Claude (Anthropic) and ChatGPT (OpenAI). Initially we investigated the weak form of the conjecture posed in [6]. We discretized the problem and considered polygonal curves with 6 sides. The AI models found a counterexample of this form, which we then smoothed and extended to a belt, and after a perturbation to an annulus by numerical means. Later, on a separate track, we investigated the conjecture via Lelieuvre's formula, which reduced the problem to the computations above. The authors have checked all AI assisted content and take full responsibility for this manuscript.

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