A LOCAL ISOPERIMETRIC INEQUALITY FOR BALLS WITH NONPOSITIVE CURVATURE

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Abstract. We show that small perturbations of the metric of a ball in Euclidean n-space to metrics with nonpositive curvature do not reduce the isoperimetric ratio. Furthermore, the isoperimetric ratio is preserved only if the perturbation corresponds to a homothety of the ball. These results establish a sharp local version of the Cartan-Hadamard conjecture.

1. Introduction

The *isoperimetric ratio* of a compact Riemannian n-manifold Ω with boundary $\partial\Omega$ is given by $\mathrm{I}(\Omega) := |\partial\Omega|^n/|\Omega|^{n-1}$, where $|\Omega|$ denotes the volume and $|\partial\Omega|$ the perimeter of Ω. The *Cartan-Hadamard conjecture* [4,10,19] states that if Ω forms a domain in a complete simply connected manifold of nonpositive (sectional) curvature, known as a *Cartan-Hadamard manifold*, then

$$\mathrm{I}(\Omega) \geqslant \mathrm{I}(B^n_\delta),$$

where $B^n_\delta := (B^n, \delta)$ is the closed unit ball $B^n \subset \mathbf{R}^n$ endowed with the Euclidean metric δ. Furthermore, $\mathrm{I}(\Omega) = \mathrm{I}(B^n_\delta)$ only if Ω is isometric to a Euclidean ball. Here we show that the conjecture holds in a local sense. Let $\mathcal{M}_0(B^n)$ be the space of $\mathcal{C}^\infty$ nonpositively curved metrics $g = (g_{ij})$ on B^n with the $\mathcal{C}^2$-norm $|g|_{\mathcal{C}^2(B^n)} := \sup_{ij} |g_{ij}|_{\mathcal{C}^2(B^n)}$, and B^n_g denote the corresponding Riemannian manifolds.

Theorem 1.1. *There exists $\varepsilon > 0$ such that for all metrics $g \in \mathcal{M}_0(B^n)$ with $|g-\delta|_{\mathcal{C}^2(B^n)} \leqslant \varepsilon$, $\mathrm{I}(B^n_g) \geqslant \mathrm{I}(B^n_\delta)$. Furthermore, $\mathrm{I}(B^n_g) = \mathrm{I}(B^n_\delta)$ only if B^n_g is isometric to a Euclidean ball.*

To establish this result we show that, for small ε, B^n_g is isometric via normal coordinates to a star-shaped domain $\Omega \subset \mathbf{R}^n$ with metric $\overline{g}$, denoted $\Omega_{\overline{g}} := (\Omega, \overline{g})$. So $\partial\Omega_{\overline{g}}$ is the graph of a radial function f on the unit sphere S^{n-1} in the Euclidean sense. Using Rauch's comparison theorem, we find a general inequality (4) for $\mathrm{I}(\Omega_{\overline{g}})$ in terms of f and the Jacobian of the exponential map of $\Omega_{\overline{g}}$ which may be of independent interest. It follows that $\mathrm{I}(\Omega_{\overline{g}}) \geqslant \mathrm{I}(\Omega_\delta)$ for small ε, via a variational technique that we devise below. But $\mathrm{I}(\Omega_\delta) \geqslant \mathrm{I}(B^n_\delta)$, by the classical isoperimetric inequality in $\mathbf{R}^n$, which completes the proof. Refining this method, we generalize Theorem 1.1 to metrics with curvature $\leqslant k \leqslant 0$, as described in Theorem 3.1.

The Cartan-Hadamard conjecture, which would extend the classical isoperimetric inequality [12,27], is known to hold only in dimensions $\leqslant 4$ [13,21,29]. Some partial results are also known in higher dimensions for geodesic balls [5], small volumes

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[14, 25, 26], large volumes [11, 30], hyperbolic space [7, 28], or with a dimension-dependent constant [13, 20]. In contrast to Theorem 1.1, the results for small and large volumes all require a negative upper bound for curvature. See [17, 18, 22, 24] for some more recent studies, [15] for an introduction to the problem, and [16] for applications to Sobolev inequalities.

Note 1.2. The condition in Theorem 1.1 that the metric g be close to the Euclidean metric δ cannot be omitted. Indeed there exist negatively curved metrics g on the unit ball B^3 such that $I(B^3_g)$ is arbitrarily small [22, Cor. 1.10]. These examples also show that nonpositively curved balls cannot in general be embedded isometrically in a Cartan-Hadamard manifold (since the Cartan-Hadamard conjecture holds in dimension 3). Thus while Theorem 1.1 adds more credence to the Cartan-Hadamard conjecture, it will not be subsumed by a positive resolution of it.

Note 1.3. Theorem 1.1 does not generalize to geodesic balls in the hyperbolic space $\mathbf{H}^n_k$, of constant curvature $k < 0$, since the isoperimetric ratio of balls in $\mathbf{H}^n_k$ is an increasing function of their radius. More precisely, consider the geodesic balls $U(r)$ of radius $r \leqslant 1$ centered at a point o of $\mathbf{H}^n_k$. There are natural diffeomorphisms $\varphi_r \colon B^n \to U(r)$ given by rescalings of normal coordinates centered at o. Let g_r be the corresponding pullback metrics. Then g_r has constant curvature k, and $g_r \to g_1$ in the $\mathcal{C}^2$-topology, as $r \to 1^-$, but $\mathrm{I}(B^n_{g_r}) < \mathrm{I}(B^n_{g_1})$ for $r < 1$.

2. Preliminaries

Let $\mathcal{M}_k(B^n)$ be the space of $\mathcal{C}^\infty$ metrics g on B^n with curvature bounded above by $k \in (-\infty, 0]$. For each $x \in B^n$ we identify g_x with its matrix representation $(g_{ij}(x))$ with respect to the standard basis e_i of $\mathbf{R}^n$. So

$$g_x(v, w) = v^T g_x w, \tag{1}$$

for tangent vectors v, $w \in T_x B^n_g \simeq \mathbf{R}^n$. The $\mathcal{C}^\ell$-*topology* on $\mathcal{M}_k(B^n)$ is induced by the norm $|g|_{\mathcal{C}^\ell(B^n)} := \sup_{ij} |g_{ij}|_{\mathcal{C}^\ell(B^n)}$. Here we record some basic observations on the structure of $\mathcal{M}_k(B^n)$, and its representation in normal coordinates.

We say B^n_g is *strictly convex* provided that every pair of its points can be joined by a unique geodesic, and the second fundamental form of ∂B^n_g with respect to the outward normal is positive definite. We need the following fact whose proof utilizes the theory of CAT(0) spaces [1, 8, 9], which are generalizations of Cartan-Hadamard manifolds. More precisely, a CAT(0) space is a metric space where every pair of points may be joined by a unique curve realizing the distance between the points, and the curvature is nonpositive in the sense of Alexandrov.

Lemma 2.1. *The set of metrics $g \in \mathcal{M}_k(B^n)$ such that B^n_g is strictly convex is open in the $\mathcal{C}^1$-topology.*

Proof. Fix a metric $g_0 \in \mathcal{M}_k(B^n)$ such that $B^n_{g_0}$ is strictly convex. Consider metrics $g \in \mathcal{M}_k(B^n)$ with $|g - g_0|_{\mathcal{C}^1(B^n)} \leqslant \varepsilon$, for a constant $\varepsilon \in (0, \infty)$. The second fundamental form of ∂B^n_g, with respect to the outward normal ν, is given by $\mathrm{II}_g(v, w) := g(D^g_v \nu, w)$, where D^g is the covariant derivative with respect to g, and

v, w are tangent vectors of ∂B^n_g. Note that ∂B^n_g is a level set of the function $F(x) := |x|$, where $|\cdot|$ is the Euclidean norm. Then $\nu(x) = \nabla^g F(x)/|\nabla^g F(x)|_g$, where ∇^g is the gradient with respect to g, and $|\cdot|_g := \sqrt{g(\cdot,\cdot)}$. Furthermore, D^g is determined by the Christoffel symbols, which depend on the first derivatives of g. Hence $g \mapsto \mathrm{I\!I}_g$ is continuous in the $\mathcal{C}^1$-topology. So, for sufficiently small ε, $\mathrm{I\!I}_g$ remains positive definite, since $\mathrm{I\!I}_{g_0}$ is positive definite.

Since $\mathrm{I\!I}_g$ is positive definite and g has nonpositive curvature, the curvature of B^n_g is bounded above by 0 in Alexandrov's sense [2, p. 704]. So B^n_g is locally a CAT(0) space. Then, since B^n_g is simply connected, it is a CAT(0) space by the generalized Cartan-Hadamard theorem [3, 8, 9]. In particular, every pair of points in B^n_g may be joined by a unique shortest curve γ. Since $\mathrm{I\!I}_g$ is positive definite, the interior of γ cannot touch ∂B^n_g. Hence γ is a (Riemannian) geodesic. So every pair of points of B^n_g can be joined by a unique minimal geodesic. It follows that the cut locus of every point of B^n_g is empty, so these geodesics are unique. □

Let $\mathcal{M}^\star_k(B^n) \subset \mathcal{M}_k(B^n)$ consist of metrics such that B^n_g is *star-shaped* (with respect to its center o), i.e., there exists a domain $\overline{B^n_g}$ in the tangent space $T_oB^n_g \simeq \mathbf{R}^n$ such that the exponential map $\exp_o\colon \overline{B^n_g} \to B^n_g$ is a diffeomorphism, and the radial geodesics of B^n_g, which emanate from o, meet ∂B^n_g transversely. In particular, if B^n_g is strictly convex then it is star-shaped.

Since g is symmetric and positive definite, there exists a positive definite symmetric matrix $h := \sqrt{g_o}$. Then h^{-1} also exists and is positive definite. Let $\overline{e}_i := h^{-1}e_i$. Then $g_o(\overline{e}_i, \overline{e}_j) = e_i^T(h^{-1})^T h^2 h^{-1} e_j = \delta(e_i, e_j)$ by (1). So $\overline{e}_i$ form an orthonormal basis for $T_oB^n_g$, which depends continuously on g. Let $\varphi = \varphi_g\colon \mathbf{R}^n \to T_oB^n_g$ be the corresponding coordinate map given by $\varphi(x) := \sum x_i\overline{e}_i$. Set $\Omega := \varphi^{-1}(\overline{B}^n_g)$, and $\widetilde{\exp}_o := \exp_o \circ \varphi|_\Omega$. Then $\widetilde{\exp}_o\colon \Omega \to B^n$ is a diffeomorphism. Let $\overline{g} := \widetilde{\exp}_o^*(g)$ be the pullback metric. Then $\widetilde{\exp}_o\colon \Omega_{\overline{g}} \to B^n_g$ is an isometry. Furthermore, the standard coordinates x_i of $\mathbf{R}^n$ form normal coordinates on $\Omega_{\overline{g}}$. So

$$\overline{g}_o(e_i, e_j) = g_o\big(d(\widetilde{\exp}_o)_o(e_i), d(\widetilde{\exp}_o)_o(e_j)\big) = g_o(\overline{e}_i, \overline{e}_j) = \delta(e_i, e_j). \tag{2}$$

For $\theta \in S^{n-1}$, let ρ_θ be the radial geodesic which connects o to $\partial\Omega_{\overline{g}}$ with initial direction θ. Note that ρ_θ intersects $\partial\Omega_{\overline{g}}$ transversely, since the corresponding radial geodesics $\widetilde{\exp}_o(\rho_\theta)$ of B^n_g are transversal to ∂B^n_g by definition. Let $f_{\overline{g}}(\theta) := \mathrm{Length}(\rho_\theta)$ be the *radial function* of $\partial\Omega_{\overline{g}}$. Since x_i are normal coordinates, $\rho_\theta(t) = t\theta$ for $0 \leqslant t \leqslant f_{\overline{g}}(\theta)$. So $\partial\Omega_{\overline{g}}$ is the graph of $f_{\overline{g}}$ over S^{n-1} in the Euclidean sense. Note that $\partial\Omega_{\overline{g}}$ is a $\mathcal{C}^\infty$ hypersurface, since $\partial\Omega_{\overline{g}} = (\widetilde{\exp}_o)^{-1}(\partial B^n_g)$. Thus $f_{\overline{g}}$ is $\mathcal{C}^\infty$, since ρ_θ are transversal to $\partial\Omega_{\overline{g}}$.

Lemma 2.2. *The mapping $\mathcal{M}^\star_k(B^n) \ni g \mapsto f_{\overline{g}} \in \mathcal{C}^1(S^{n-1})$ is continuous in the $\mathcal{C}^2$-topology.*

Proof. Fix $g \in \mathcal{M}^\star_k(B^n)$ and let $g_i \in \mathcal{M}^\star_k(B^n)$ be a sequence with $g_i \to g$ in the $\mathcal{C}^2$-topology. Let $\exp_o$ and $\exp^i_o$ be the exponential maps of B^n_g and $B^n_{g_i}$ respectively. Then $(\exp^i_o)^{-1} \to (\exp_o)^{-1}$ in the $\mathcal{C}^1$-topology of maps from B^n to $T_oB^n \simeq \mathbf{R}^n$, because exponential maps are determined by ODE whose coefficients involve the first derivatives of the metric. Consequently,

$$\partial\Omega_{\overline{g}_i} = (\widetilde{\exp^i_o})^{-1}(\partial B^n_{g_i}) \quad\longrightarrow\quad (\widetilde{\exp}_o)^{-1}(\partial B^n_g) = \partial\Omega_{\overline{g}}$$

in the $\mathcal{C}^1$-topology of hypersurfaces embedded in $\mathbf{R}^n$, because here the differentials $d(\widetilde{\exp^i_o})^{-1} = \varphi_{g_i}^{-1} \circ d(\exp^i_o)^{-1} \to \varphi^{-1} \circ d(\exp_o)^{-1} = d(\widetilde{\exp}_o)^{-1}$ in the $\mathcal{C}^0$-topology of maps from B^n to $\mathbf{R}^n$, and so $(\widetilde{\exp^i_o})^{-1} \to \widetilde{\exp}_o^{-1}$ in the $\mathcal{C}^1$-topology. Hence the radial functions $f_{\overline{g}_i} \to f_{\overline{g}}$ in $\mathcal{C}^1(S^{n-1})$. □

3. Proof of the Main Result

Here we establish the following result, which implies Theorem 1.1 via the above lemmas. For any metric $g \in \mathcal{M}^\star_k(B^n)$, let $\Omega_{\overline{g}} \subset \mathbf{R}^n$ denote the corresponding star-shaped domain, in normal coordinates, with radial function $f_{\overline{g}}$ as defined above. For $k \leqslant 0$, let δ^k be the metric of constant curvature k on $\mathbf{R}^n$ in normal coordinates centered at o. So $\delta^0 = \delta$ is the standard Euclidean metric. Also note that Ω_{δ^k} is isometric to a domain with radial function $f_{\overline{g}}$ in $\mathbf{R}^n$ or hyperbolic space $\mathbf{H}^n_k$, if $k = 0$ or $k < 0$ respectively.

Theorem 3.1. *For every $R > 0$ there exists $\varepsilon > 0$ such that for all metrics $g \in \mathcal{M}^\star_k(B^n)$ with $|f_{\overline{g}} - R|_{\mathcal{C}^1(S^{n-1})} \leqslant \varepsilon$, $\mathrm{I}(\Omega_{\overline{g}}) \geqslant \mathrm{I}(\Omega_{\delta^k})$. Furthermore, $\mathrm{I}(\Omega_{\overline{g}}) = \mathrm{I}(\Omega_{\delta^k})$ only if $\overline{g} = \delta^k$.*

If ε in Theorem 1.1 is sufficiently small, then B^n_g is star-shaped by Lemma 2.1, so $g \in \mathcal{M}^\star_0(B^n)$. Furthermore, $|f_{\overline{g}} - 1|_{\mathcal{C}^1(S^{n-1})}$ can be made arbitrarily small by Lemma 2.2, since $f_{\overline{\delta}} = 1$. So Theorem 3.1, together with the classical isoperimetric inequality, yields that

$$\mathrm{I}(B^n_g) = \mathrm{I}(\Omega_{\overline{g}}) \geqslant \mathrm{I}(\Omega_\delta) \geqslant \mathrm{I}(B^n_\delta).$$

If $\mathrm{I}(B^n_g) = \mathrm{I}(B^n_\delta)$, then $\mathrm{I}(\Omega_{\overline{g}}) = \mathrm{I}(\Omega_\delta)$, which yields $\overline{g} = \delta$. So Theorem 1.1 indeed follows from Theorem 3.1 (this argument cannot be generalized to $k < 0$, as we had pointed out in Note 1.3, due to the use of the classical isoperimetric inequality). Next, to prove Theorem 3.1, we begin by recording some basic facts from linear algebra. For any square matrix A, let $A(v, w) := v^T A w$ denote the corresponding quadratic form.

Lemma 3.2. *Let A and B be symmetric positive definite $n \times n$ matrices. Suppose that for all $v \in \mathbf{R}^n$, $A(v, v) \geqslant B(v, v)$. Then*

(i) $\det(A) \geqslant \det(B)$,
(ii) $\det(A)A^{-1}(v, v) \geqslant \det(B)B^{-1}(v, v)$.

Furthermore, equality holds in (i), and in (ii) for all v, only if $A = B$.

Proof. Let $Q^{-1}\Lambda Q$ be the spectral decomposition of B. Then $v^T A v \geqslant v^T Q^{-1} \Lambda Q v$. Setting $w := \Lambda^{1/2} Q v$, and $M := \Lambda^{-1/2} Q A Q^{-1} \Lambda^{-1/2}$, we obtain

$$M(w, w) \geqslant |w|^2.$$

So the eigenvalues λ_i of M are $\geqslant 1$. Thus $\det(M) \geqslant 1$. But $\det(M) = \det(A)/\det(B)$. So we have (i). If equality holds in (i), then $\lambda_i = 1$, so M is the identity matrix, which yields $A = B$. Inequality (ii) is equivalent to

$$\det(M)M^{-1}(w,w) \geqslant |w|^2,$$

which we rewrite as $(\prod \lambda_i) \sum w_j^2 \lambda_j^{-1} \geqslant \sum w_j^2$. This holds since $(\prod \lambda_i)\lambda_j^{-1} \geqslant 1$, as $\lambda_i \geqslant 1$. So we obtain (ii). Equality holds in (ii) for all v only if equality holds in the above inequality for all w. Then $\lambda_i = 1$, which again yields $A = B$. □

Let $\overline{\exp}$ denote the exponential map of $\Omega_{\overline{g}}$. By (2), the natural identification $T_o\Omega_{\overline{g}} \simeq \mathbf{R}^n$ is an isometry. We also have $T_x(T_o\Omega_{\overline{g}}) \simeq T_x\mathbf{R}^n \simeq \mathbf{R}^n$. Furthermore, since x_i are normal coordinates, $\overline{\exp}_o(x) = x$ for $x \in \Omega_{\overline{g}}$. The volume element of $T_x\Omega_{\overline{g}}$ is $\sqrt{\det(\overline{g}_x)}dx$, where dx is the standard volume element of $\mathbf{R}^n$. Hence the Jacobian of $\overline{\exp}_o$ at x is given by

$$J_{\overline{g}}(x) = \sqrt{\det(\overline{g}_x)}.$$

Set $J_k := J_{\delta^k}$. Recall that $|\cdot|_g := \sqrt{g(\cdot,\cdot)}$, and set $|\cdot|_k := |\cdot|_{\delta^k}$. So $|\cdot|_0 = |\cdot|$ is the Euclidean norm. Also recall that ∇^g denotes the gradient with respect to g, and set $\nabla^k := \nabla^{\delta^k}$. So $\nabla^0 = \nabla$ is the Euclidean gradient. Note that $g(v, \nabla^g f) = df(v) = \delta(v, \nabla f)$, which yields $v^T g \nabla^g f = v^T \nabla f$. So

$$\nabla^g f = g^{-1}\nabla f. \tag{3}$$

Let $r(x) := |x|$ and $\theta(x) := x/|x|$ be the polar coordinates on $\mathbf{R}^n \setminus \{o\}$. Note that $J_k(r\theta)$ does not depend on θ. So we denote this quantity by $J_k(r)$. The last lemma together with Rauch's comparison theorem yields:

Lemma 3.3. *For any metric $g \in \mathcal{M}_k^\star(B^n)$, and differentiable function f on $\Omega_{\overline{g}}$,*

(i) *$r \mapsto J_{\overline{g}}(r\theta)/J_k(r)$ is nondecreasing,*
(ii) *$J_{\overline{g}}(r\theta) \geqslant J_k(r)$, with equality everywhere only if $\overline{g} = \delta^k$,*
(iii) *$J_{\overline{g}}(r\theta)\, |\nabla^{\overline{g}} f(r\theta)|_{\overline{g}} \geqslant J_k(r)\, |\nabla^k f(r\theta)|_k$.*

Proof. For (i) see [11, 33.1.6] or [6, p. 253]. Inequality (ii) follows from the inequality

$$\overline{g}(v,v) \geqslant \delta^k(v,v),$$

which is due to Jacobi's equation in normal coordinates [23, Thm. 11.10], and Lemma 3.2(i). Inequality (iii) also follows quickly from the above inequality via Lemma 3.2(ii) and (3). □

Now we are ready to establish our main result.

Proof of Theorem 3.1. Set $f = f_{\overline{g}}$, and define

$$\rho(\theta) := \frac{J_{\overline{g}}\big(f(\theta)\theta\big)}{J_k\big(f(\theta)\big)}, \qquad \mathcal{J}_k(s) := \int_0^s J_k(r) r^{n-1} dr.$$

By Lemma 3.3(i), $J_{\overline{g}}(r\theta) \leqslant \rho(\theta) J_k(r)$ for $r \leqslant f(\theta)$. Thus the volume

$$|\Omega_{\overline{g}}| = \int_{\Omega} J_{\overline{g}}(x)dx = \int_{S^{n-1}} \int_0^{f(\theta)} J_{\overline{g}}(r\theta) r^{n-1} dr d\theta$$
$$\leqslant \int_{S^{n-1}} \int_0^{f(\theta)} \rho(\theta) J_k\big(r\big) r^{n-1} dr d\theta = \int_{S^{n-1}} \rho(\theta) \mathcal{J}_k\big(f(\theta)\big) d\theta,$$

where $d\theta$ is the standard volume element of S^{n-1}. Extend f radially to $\Omega \setminus \{o\}$ by setting $f(x) := f(\theta(x))$, and let $F(x) := r(x) - f(x)$. Then $|\nabla^{\overline{g}} F|^2_{\overline{g}} = 1 + |\nabla^{\overline{g}} f|^2_{\overline{g}}$ since $|\nabla^{\overline{g}} r|_{\overline{g}} = |\nabla r| = 1$, and by the Gauss lemma $\overline{g}(\nabla^{\overline{g}} r, \nabla^{\overline{g}} f) = 0$. Thus, by the coarea formula, the perimeter

$$|\partial \Omega_{\overline{g}}| = |F^{-1}(0)| = \frac{d}{ds}\Big|_{s=0} \int_{-s}^{0} \big|F^{-1}(t)\big|\, dt = \frac{d}{ds}\Big|_{s=0} \int_{F^{-1}([-s,0])} \big|\nabla^{\overline{g}} F(x)\big|_{\overline{g}}\ J_{\overline{g}}(x) dx$$
$$= \frac{d}{ds}\Big|_{s=0} \int_{S^{n-1}} \int_{f(\theta)-s}^{f(\theta)} \big|\nabla^{\overline{g}} F(r\theta)\big|_{\overline{g}}\ J_{\overline{g}}(r\theta) r^{n-1} dr d\theta$$
$$= \int_{S^{n-1}} \sqrt{1 + |\nabla^{\overline{g}} f(\theta)|^2_{\overline{g}}}\ J_{\overline{g}}\big(f(\theta)\theta\big) f^{n-1}(\theta)\, d\theta.$$

Now applying Lemma 3.3(iii), we obtain

$$\mathrm{I}(\Omega_{\overline{g}}) \geqslant \frac{\left(\int_{S^{n-1}} \sqrt{\rho^2 + |\nabla^k f|^2_k}\ J_k(f) f^{n-1} d\theta\right)^n}{\left(\int_{S^{n-1}} \rho\, \mathcal{J}_k(f) d\theta\right)^{n-1}}. \tag{4}$$

So we may write

$$\mathrm{I}(\Omega_{\overline{g}}) \geqslant \lambda^{n-1}(1), \qquad \text{where} \qquad \lambda(t) := \frac{\left(\int_{S^{n-1}} \sqrt{\rho^{2t} + |\nabla^k f|^2_k}\, J_k(f) f^{n-1} d\theta\right)^{\frac{n}{n-1}}}{\int_{S^{n-1}} \rho^t \mathcal{J}_k(f) d\theta},$$

for $t \in [0,1]$. Let $A^{\frac{n}{n-1}}$ denote the numerator and B the denominator of $\lambda(t)$. Then

$$\lambda'(t) = \frac{A^{\frac{1}{n-1}}}{B}\left(\frac{n}{n-1} A' - \frac{A}{B} B'\right) = \frac{A^{\frac{1}{n-1}}}{B} \int_{S^{n-1}} C \ln(\rho) \rho^t J_k(f) f^{n-1}\, d\theta, \tag{5}$$

where

$$C := \frac{n}{n-1} \frac{\rho^t}{\sqrt{\rho^{2t} + |\nabla^k f|^2_k}} - \frac{A}{B} \frac{\mathcal{J}_k(f)}{J_k(f) f^{n-1}}.$$

By Lemma 3.3(ii), $\rho \geqslant 1$. Thus the sign of λ' depends on that of C. By assumption, $|f - R| \leqslant \varepsilon$ and $|\nabla f| \leqslant \varepsilon$. So by (3) and continuity of J_k and $\mathcal{J}_k$, for any $\overline{\varepsilon} > 0$ we may choose ε so small that

$$|f - R| \leqslant \overline{\varepsilon}, \qquad |\nabla^k f|_k \leqslant \overline{\varepsilon}, \qquad \left|\frac{J_k(f)}{J_k(R)} - 1\right| \leqslant \overline{\varepsilon}, \quad \text{and} \quad \left|\frac{\mathcal{J}_k(f)}{\mathcal{J}_k(R)} - 1\right| \leqslant \overline{\varepsilon}.$$

Then

$$A \leqslant \sqrt{1+\bar{\varepsilon}^2} J_k(R)\,(1+\bar{\varepsilon})\,(R+\bar{\varepsilon})^{n-1} \int_{S^{n-1}} \rho^t d\theta,$$

$$B \geqslant \mathcal{J}_k(R)(1-\bar{\varepsilon}) \int_{S^{n-1}} \rho^t d\theta.$$

Thus

$$C \geqslant \frac{n}{n-1} \frac{1}{\sqrt{1+\bar{\varepsilon}^2}} - \left(\frac{1+\bar{\varepsilon}}{1-\bar{\varepsilon}}\right)^2 \left(\frac{R+\bar{\varepsilon}}{R-\bar{\varepsilon}}\right)^{n-1} \sqrt{1+\bar{\varepsilon}^2}.$$

So if $\bar{\varepsilon}$ is sufficiently small, then $C > 0$, which yields $\lambda' \geqslant 0$. Thus $\lambda(1) \geqslant \lambda(0)$. But $\lambda^{n-1}(0) = \mathrm{I}(\Omega_{\delta^k})$. So $\mathrm{I}(\Omega_{\bar{g}}) \geqslant \mathrm{I}(\Omega_{\delta^k})$ as desired.

Finally suppose that $\mathrm{I}(\Omega_{\bar{g}}) = \mathrm{I}(\Omega_{\delta^k})$. Then $\lambda(1) = \lambda(0)$. So $\lambda' = 0$ identically. Since $C > 0$, it follows from (5) that $\rho = 1$ identically. So $J_{\bar{g}}(f(\theta)\theta) = J_k(f(\theta))$. Consequently, by Lemma 3.3(i), $J_{\bar{g}}(r\theta) \leqslant J_k(r)$ for $r \leqslant f(\theta)$. Thus $\bar{g} = \delta^k$ by Lemma 3.3(ii). □

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