A LOCAL ISOPERIMETRIC INEQUALITY FOR BALLS WITH NONPOSITIVE CURVATURE

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Abstract. We show that small perturbations of the metric of a ball in Euclidean n-space to metrics with nonpositive curvature do not reduce the isoperimetric ratio. Furthermore, the isoperimetric ratio is preserved only if the perturbation corresponds to a homothety of the ball. These results establish a sharp local version of the Cartan-Hadamard conjecture.

1. Introduction

The *isoperimetric ratio* of a compact Riemannian n-manifold Ω with boundary $\partial\Omega$ is given by $\mathrm{I}(\Omega) := |\partial\Omega|^n/|\Omega|^{n-1}$, where $|\Omega|$ denotes the volume and $|\partial\Omega|$ the perimeter of Ω. The *Cartan-Hadamard conjecture* [4, 10, 18] states that if Ω forms a domain in a complete simply connected manifold of nonpositive curvature, known as a *Cartan-Hadamard manifold*, then

$$\mathrm{I}(\Omega) \geqslant \mathrm{I}(B^n_\delta),$$

where B^n_δ denotes the unit ball $B^n \subset \mathbf{R}^n$ endowed with the Euclidean metric δ. Furthermore, $\mathrm{I}(\Omega) = \mathrm{I}(B^n_\delta)$ only if Ω is isometric to a Euclidean ball. Here we show that the conjecture holds in a local sense. Let $\mathcal{M}_0(B^n)$ be the space of $\mathcal{C}^\infty$ nonpositively curved metrics $g = (g_{ij})$ on B^n with the $\mathcal{C}^2$-norm $|g|_{\mathcal{C}^2(B^n)} := \sup_{ij} |g_{ij}|_{\mathcal{C}^2(B^n)}$, and B^n_g denote the corresponding Riemannian manifolds.

Theorem 1.1. *There exists $\varepsilon > 0$ such that for all metrics $g \in \mathcal{M}_0(B^n)$ with $|g-\delta|_{\mathcal{C}^2(B^n)} \leqslant \varepsilon$, $\mathrm{I}(B^n_g) \geqslant \mathrm{I}(B^n_\delta)$. Furthermore, $\mathrm{I}(B^n_g) = \mathrm{I}(B^n_\delta)$ only if B^n_g is isometric to a Euclidean ball.*

To establish this result we show that, for small ε, B^n_g is isometric via normal coordinates to a star-shaped domain $\Omega \subset \mathbf{R}^n$ with metric $\overline{g}$, denoted by $\Omega_{\overline{g}}$. So $\partial\Omega_{\overline{g}}$ is the graph of a radial function f on the unit sphere S^{n-1}, with $|f-1|_{\mathcal{C}^1(S^{n-1})} \to 0$ as $|g-\delta|_{\mathcal{C}^2(B^n)} \to 0$. Using Rauch's comparison theorem, we bound $\mathrm{I}(\Omega_{\overline{g}})$ from below in terms of f and the Jacobian J of the exponential map of $\Omega_{\overline{g}}$. Since $\overline{g}$ has nonpositive curvature, $J \geqslant 1$ with equality only if $\overline{g} = \delta$. It follows that $\mathrm{I}(\Omega_{\overline{g}}) \geqslant \mathrm{I}(\Omega_\delta)$ for small ε, via a variational technique. But $\mathrm{I}(\Omega_\delta) \geqslant \mathrm{I}(B^n_\delta)$, by the classical isoperimetric inequality in $\mathbf{R}^n$, which completes the proof. Refining this method, we generalize Theorem 1.1 to metrics with curvature $\leqslant k \leqslant 0$, as described in Theorem 3.1.

The Cartan-Hadamard conjecture, which would extend the classical isoperimetric inequality [12, 26], is known to hold only in dimensions $\leqslant 4$ [13, 20, 28]. Some partial results are also known in higher dimensions for geodesic balls [5], small volumes

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[14, 24, 25], large volumes [11, 29], hyperbolic space [7, 27], and up to a constant [13, 19]. See [16, 17, 21, 23] for some more recent studies and [15] for an introduction to the problem.

Note 1.2. Theorem 1.1 does not generalize to geodesic balls in the hyperbolic space $\mathbf{H}^n_k$, of constant curvature $k<0$, since the isoperimetric ratio of balls in $\mathbf{H}^n_k$ is an increasing function of their radius. More precisely, consider the geodesic balls $U(r)$ of radius $r \leqslant 1$ centered at a point o of $\mathbf{H}^n_k$. There are natural diffeomorphisms $\varphi_r \colon B^n \to U(r)$ given by rescalings of normal coordinates centered at o. Let g_r be the corresponding pullback metrics. Then g_r has constant curvature k, and $g_r \to g_1$ in the $\mathcal{C}^2$-topology, but $\mathrm{I}(B^n_{g_r}) < \mathrm{I}(B^n_{g_1})$ for $r<1$.

2. Preliminaries

Let $\mathcal{M}_k(B^n)$ be the space of $\mathcal{C}^\infty$ metrics g on B^n with curvature bounded above by $k \in (-\infty, 0]$. For each $x \in B^n$ we identify g_x with its matrix representation $(g_{ij}(x))$ with respect to the standard basis e_i of $\mathbf{R}^n$. So

$$g_x(v,w) = v^T g_x w, \tag{1}$$

for tangent vectors v, $w \in T_x B^n_g \simeq \mathbf{R}^n$. The *$\mathcal{C}^\ell$-topology* on $\mathcal{M}_k(B^n)$ is induced by the norm $|g|_{\mathcal{C}^\ell(B^n)} := \sup_{ij} |g_{ij}|_{\mathcal{C}^\ell(B^n)}$. Here we record some basic observations on the structure of $\mathcal{M}_k(B^n)$, and its representation in normal coordinates.

We say B^n_g is *strictly convex* provided that every pair of its points can be joined by a unique geodesic, and the second fundamental form of ∂B^n_g with respect to the outward normal is positive definite. We need the following fact whose proof utilizes the theory of CAT(0) spaces [1, 8, 9], which are generalizations of Cartan-Hadamard manifolds. More precisely, a CAT(0) space is a metric space where every pair of points may be joined by a unique curve realizing the distance between the points, and the curvature is nonpositive in the sense of Alexandrov.

Lemma 2.1. *The set of metrics $g \in \mathcal{M}_k(B^n)$ such that B^n_g is strictly convex is open in the $\mathcal{C}^1$-topology.*

Proof. Fix a metric $g_0 \in \mathcal{M}_k(B^n)$ such that $B^n_{g_0}$ is strictly convex. Consider metrics $g \in \mathcal{M}_k(B^n)$ with $|g-g_0|_{\mathcal{C}^1(B^n)} \leqslant \varepsilon$. The second fundamental form of ∂B^n_g, with respect to the outward normal ν, is given by $\mathrm{I\!I}_g(v,w) := g(D^g_v \nu, w)$, where D^g is the covariant derivative with respect to g, and v, w are tangent vectors of ∂B^n_g. Note that ∂B^n_g is a level set of the function $F(x) := |x|$, where $|\cdot|$ is the Euclidean norm. Then $\nu(x) = \nabla^g F(x)/|\nabla^g F(x)|_g$, where ∇^g is the gradient with respect to g, and $|\cdot|_g := \sqrt{g(\cdot,\cdot)}$. Furthermore, D^g is determined by the Christoffel symbols, which depend on the first derivatives of g. Hence $g \mapsto \mathrm{I\!I}_g$ is continuous in the $\mathcal{C}^1$-topology. So, for sufficiently small ε, $\mathrm{I\!I}_g$ remains positive definite, since $\mathrm{I\!I}_{g_0}$ is positive definite.

Since $\mathrm{I\!I}_g$ is positive definite and g has nonpositive curvature, the curvature of B^n_g is bounded above by 0 in Alexandrov's sense [2, p. 704]. So B^n_g is locally a CAT(0) space. Then, since B^n_g is simply connected, it is a CAT(0) space by the generalized Cartan-Hadamard theorem [3, 8, 9]. In particular, every pair of points in B^n_g may be joined by a unique shortest curve γ. Since II_g is positive definite, the interior of γ cannot touch ∂B^n_g. Hence γ is a (Riemannian) geodesic. So every pair of points of B^n_g can be joined by a unique minimal geodesic. It follows that the cut locus of every point of B^n_g is empty, so these geodesics are unique. $\square$

Let $\mathcal{M}^\star_k(B^n) \subset \mathcal{M}_k(B^n)$ consist of metrics such that B^n_g is *star-shaped* (with respect to its center o), i.e., there exists a domain $\overline{B^n_g}$ in the tangent space $T_oB^n_g \simeq \mathbf{R}^n$ such that the exponential map $\exp_o\colon \overline{B^n_g} \to B^n_g$ is a diffeomorphism, and the radial geodesics of B^n_g, which emanate from o, meet ∂B^n_g transversely. In particular, if B^n_g is strictly convex then it is star-shaped.

Since g is symmetric and positive definite, there exists a symmetric matrix $h := \sqrt{g_o}$. Let $\overline{e}_i := h^{-1}e_i$. Then $g_o(\overline{e}_i, \overline{e}_j) = e_i^T(h^{-1})^T h^2 h^{-1} e_j = \delta(e_i, e_j)$ by (1). So $\overline{e}_i$ form an orthonormal basis for $T_oB^n_g$, which depends continuously on g. Let $\varphi = \varphi_g \colon \mathbf{R}^n \to T_oB^n_g$ be the corresponding coordinate map given by $\varphi(x) := \sum x_i\overline{e}_i$. Set $\Omega := \varphi^{-1}(\overline{B}^n_g)$, and $\widetilde{\exp}_o := \exp_o \circ\, \varphi|_\Omega$. Then $\widetilde{\exp}_o\colon \Omega \to B^n$ is a diffeomorphism. Let $\overline{g} := \widetilde{\exp}_o^*(g)$ be the pullback metric. Then $\widetilde{\exp}_o\colon \Omega_{\overline{g}} \to B^n_g$ is an isometry. Furthermore, the standard coordinates x_i of $\mathbf{R}^n$ form normal coordinates on $\Omega_{\overline{g}}$. So

$$\overline{g}_o(e_i, e_j) = g_o\big(d(\widetilde{\exp}_o)_o(e_i), d(\widetilde{\exp}_o)_o(e_j)\big) = g_o(\overline{e}_i, \overline{e}_j) = \delta(e_i, e_j). \tag{2}$$

For $\theta \in S^{n-1}$, let ρ_θ be the radial geodesic which connects o to $\partial\Omega_{\overline{g}}$ with initial direction θ. Note that ρ_θ intersect $\partial\Omega_{\overline{g}}$ transversely, since the corresponding radial geodesics $\widetilde{\exp}_o(\rho_\theta)$ of B^n_g are transversal to ∂B^n_g by definition. Let $f_{\overline{g}}(\theta) := \mathrm{Length}(\rho_\theta)$ be the *radial function* of $\partial\Omega_{\overline{g}}$. Since x_i are normal coordinates, $\rho_\theta(t) = t\theta$ for $0 \leqslant t \leqslant f(\theta)$. So $\partial\Omega_{\overline{g}}$ is the graph of $f_{\overline{g}}$ over S^{n-1} in the Euclidean sense. Note that $\partial\Omega_{\overline{g}}$ is a $\mathcal{C}^\infty$ hypersurface, since $\partial\Omega_{\overline{g}} = (\widetilde{\exp}_o)^{-1}(\partial B^n_g)$. Thus $f_{\overline{g}}$ is $\mathcal{C}^\infty$, since ρ_θ are transversal to $\partial\Omega_{\overline{g}}$.

Lemma 2.2. *The mapping $\mathcal{M}^\star_k(B^n) \ni g \mapsto f_{\overline{g}} \in \mathcal{C}^1(S^{n-1})$ is continuous in the $\mathcal{C}^2$-topology.*

Proof. Fix $g \in \mathcal{M}^\star_k(B^n)$ and let $g_i \in \mathcal{M}^\star_k(B^n)$ be a sequence with $g_i \to g$ in the $\mathcal{C}^2$-topology. Let $\exp_o$ and $\exp^i_o$ be the exponential maps of B^n_g and $B^n_{g_i}$ respectively. Then $(\exp^i_o)^{-1} \to (\exp_o)^{-1}$ in the $\mathcal{C}^1$-topology of maps from B^n to $T_oB^n \simeq \mathbf{R}^n$; because, exponential maps are determined by ODE whose coefficients involve the first derivatives of the metric. Consequently,

$$\partial\Omega_{\overline{g}_i} = (\widetilde{\exp^i_o})^{-1}(\partial B^n_{g_i}) \quad\longrightarrow\quad (\widetilde{\exp}_o)^{-1}(\partial B^n_g) = \partial\Omega_{\overline{g}}$$

in the $\mathcal{C}^1$-topology of hypersurfaces embedded in $\mathbf{R}^n$; because, the differentials $d(\widetilde{\exp^i_o})^{-1} = \varphi_{g_i}^{-1} \circ d(\exp^i_o)^{-1} \to \varphi^{-1} \circ d(\exp_o)^{-1} = d(\widetilde{\exp}_o)^{-1}$ in the $\mathcal{C}^0$-topology of maps from B^n to $\mathbf{R}^n$, and so $(\widetilde{\exp}^i_o)^{-1} \to \widetilde{\exp}_o^{-1}$ in the $\mathcal{C}^1$-topology. Hence the radial functions $f_{\overline{g}_i} \to f_{\overline{g}}$ in $\mathcal{C}^1(S^{n-1})$. $\square$

3. Proof of the Main Result

Here we establish the following result, which implies Theorem 1.1 via the above lemmas. For any metric $g \in \mathcal{M}^\star_k(B^n)$, let $\Omega_{\overline{g}} \subset \mathbf{R}^n$ denote the corresponding star-shaped domain, in normal coordinates, with radial function $f_{\overline{g}}$ as defined above. For $k \leqslant 0$, let δ^k be the metric of constant curvature k on $\mathbf{R}^n$ in normal coordinates centered at o. So $\delta^0 = \delta$ is the standard Euclidean metric. Also note that Ω_{δ^k} is isometric to a domain with radial function $f_{\overline{g}}$ in $\mathbf{R}^n$ or hyperbolic space $\mathbf{H}^n_k$, if $k = 0$ or < 0 respectively.

Theorem 3.1. *For every $R > 0$ there exists $\varepsilon > 0$ such that for all metrics $g \in \mathcal{M}^\star_k(B^n)$ with $|f_{\overline{g}} - R|_{\mathcal{C}^1(S^{n-1})} \leqslant \varepsilon$, $\mathrm{I}(\Omega_{\overline{g}}) \geqslant \mathrm{I}(\Omega_{\delta^k})$. Furthermore, $\mathrm{I}(\Omega_{\overline{g}}) = \mathrm{I}(\Omega_{\delta^k})$ only if $\overline{g} = \delta^k$.*

If ε in Theorem 1.1 is sufficiently small, then B^n_g is star-shaped by Lemma 2.1, or $g \in \mathcal{M}^\star_0(B^n)$. Furthermore, $|f_{\overline{g}} - 1|_{\mathcal{C}^1(S^{n-1})}$ can be made arbitrarily small by Lemma 2.2, since $f_{\overline{\delta}} = 1$. So Theorem 3.1, together with the classical isoperimetric inequality, yields that

$$\mathrm{I}(B^n_g) = \mathrm{I}(\Omega_{\overline{g}}) \geqslant \mathrm{I}(\Omega_\delta) \geqslant \mathrm{I}(B^n_\delta).$$

If $\mathrm{I}(B^n_g) = \mathrm{I}(B^n_\delta)$, then $\mathrm{I}(\Omega_{\overline{g}}) = \mathrm{I}(\Omega_\delta)$, which yields $\overline{g} = \delta$. So Theorem 1.1 indeed follows from Theorem 3.1. Next, to prove Theorem 3.1, we begin by recording some basic facts from linear algebra. For any square matrix A, let $A(v, w) := v^T A w$ denote the corresponding quadratic form.

Lemma 3.2. *Let A and B be symmetric positive definite $n \times n$ matrices. Suppose that for all $v \in \mathbf{R}^n$, $A(v, v) \geqslant B(v, v)$. Then*

(i) $\det(A) \geqslant \det(B)$,
(ii) $\det(A)A^{-1}(v, v) \geqslant \det(B)B^{-1}(v, v)$.

Furthermore, equality holds in (1), and in (2) for all v, only if $A = B$.

Proof. Let $Q^{-1}\Lambda Q$ be the spectral decomposition of B. Then $v^T A v \geqslant v^T Q^{-1}\Lambda Q v$. Setting $w := \Lambda^{1/2} Q v$, and $M := \Lambda^{-1/2} Q A Q^{-1} \Lambda^{-1/2}$, we obtain

$$M(w, w) \geqslant |w|^2.$$

So the eigenvalues λ_i of M are $\geqslant 1$. Thus $\det(M) \geqslant 1$. But $\det(M) = \det(A)/\det(B)$. So we have (i). If equality holds in (i), then $\lambda_i = 1$, or M is the identity matrix, which yields $A = B$. Inequality (ii) is equivalent to

$$\det(M)M^{-1}(w, w) \geqslant |w|^2,$$

which we rewrite as $(\prod \lambda_i) \sum w_j^2 \lambda_j^{-1} \geqslant \sum w_j^2$. This holds since $(\prod \lambda_i)\lambda_j^{-1} \geqslant 1$, as $\lambda_i \geqslant 1$. So we obtain (ii). Equality holds in (ii) for all v only if equality holds in the above inequality for all w. Then $\lambda_i = 1$, which again yields $A = B$. □

Let $\overline{\exp}$ denote the exponential map of $\Omega_{\overline{g}}$. By (2), the natural identification $T_o\Omega_{\overline{g}} \simeq \mathbf{R}^n$ is an isometry. We also have $T_x(T_o\Omega_{\overline{g}}) \simeq T_x\mathbf{R}^n \simeq \mathbf{R}^n$. Furthermore, since x_i are normal coordinates, $\overline{\exp}_o(x) = x$ for $x \in \Omega_{\overline{g}}$. The volume element of

$T_x\Omega_{\overline{g}}$ is $\sqrt{\det(\overline{g}_x)}dx$, where dx is the standard volume element of $\mathbf{R}^n$. Hence the Jacobian of $\overline{\exp}_o$ at x is given by

$$J_{\overline{g}}(x) = \sqrt{\det(\overline{g}_x)}.$$

Set $J_k := J_{\delta^k}$. Recall that $|\cdot|_g := \sqrt{g(\cdot,\cdot)}$, and set $|\cdot|_k := |\cdot|_{\delta^k}$. So $|\cdot|_0 = |\cdot|$ is the Euclidean norm. Also recall that ∇^g denotes the gradient with respect to g, and set $\nabla^k := \nabla^{\delta^k}$. So $\nabla^0 = \nabla$ is the Euclidean gradient. Note that $g(v, \nabla^g f) = df(v) = \delta(v, \nabla f)$, which yields $v^T g \nabla^g f = v^T \nabla f$. So

$$\nabla^g f = g^{-1}\nabla f. \tag{3}$$

Let $r(x) := |x|$ and $\theta(x) := x/|x|$ be the polar coordinates on $\mathbf{R}^n \setminus \{o\}$. Note that $J_k(r\theta)$ does not depend on θ. So we denote this quantity by $J_k(r)$. The last lemma together with Rauch's comparison theorem yields:

Lemma 3.3. *For any metric $g \in \mathcal{M}_k^\star(B^n)$, and differentiable function f on $\Omega_{\overline{g}}$,*

(i) *$r \mapsto J_{\overline{g}}(r\theta)/J_k(r)$ is nondecreasing,*
(ii) *$J_{\overline{g}}(r\theta) \geqslant J_k(r)$, with equality everywhere only if $\overline{g} = \delta^k$,*
(iii) *$J_{\overline{g}}(r\theta)\,|\nabla^{\overline{g}} f(r\theta)|_{\overline{g}} \geqslant J_k(r)\,|\nabla^k f(r\theta)|_k$.*

Proof. For (i) see [11, 33.1.6] or [6, p. 253]. Inequality (ii) follows from the inequality

$$\overline{g}(v,v) \geqslant \delta^k(v,v),$$

which is due to Jacobi's equation in normal coordinates [22, Thm. 11.10], and Lemma 3.2(i). Inequality (iii) also follows quickly from the above inequality via Lemma 3.2(ii) and (3). □

Now we are ready to establish our main result.

Proof of Theorem 3.1. Set $f = f_{\overline{g}}$, and define

$$\rho(\theta) := \frac{J_{\overline{g}}\big(f(\theta)\theta\big)}{J_k\big(f(\theta)\big)}, \qquad \mathcal{J}_k(s) := \int_0^s J_k\big(r\big) r^{n-1} dr.$$

By Lemma 3.3(i), $J_{\overline{g}}(r\theta) \leqslant \rho(\theta) J_k(r)$ for $r \leqslant f(\theta)$. Thus the volume

$$\begin{aligned} |\Omega_{\overline{g}}| = \int_\Omega J_{\overline{g}}(x)dx &= \int_{S^{n-1}} \int_0^{f(\theta)} J_{\overline{g}}(r\theta) r^{n-1} dr d\theta \\ &\leqslant \int_{S^{n-1}} \int_0^{f(\theta)} \rho(\theta) J_k\big(r\big) r^{n-1} dr d\theta = \int_{S^{n-1}} \rho(\theta)\mathcal{J}_k\big(f(\theta)\big) d\theta, \end{aligned}$$

where $d\theta$ is the standard volume element of S^{n-1}. Extend f radially to $\Omega \setminus \{o\}$ by setting $f(x) := f(\theta(x))$, and let $F(x) := r(x) - f(x)$. Then $|\nabla^{\overline{g}} F|^2_{\overline{g}} = 1 + |\nabla^{\overline{g}} f|^2_{\overline{g}}$ since $|\nabla^{\overline{g}} r|_{\overline{g}} = |\nabla r| = 1$, and by the Gauss lemma $\overline{g}(\nabla^{\overline{g}} r, \nabla^{\overline{g}} f) = 0$. Thus, by the coarea formula, the perimeter

$$
\begin{aligned}
|\partial\Omega_{\overline{g}}| = \frac{d}{ds}\Big|_{s=0}\int_{-s}^{0}\left|F^{-1}(t)\right|dt &= \frac{d}{ds}\Big|_{s=0}\int_{F^{-1}([-s,0])}\left|\nabla^{\overline{g}}F(x)\right|_{\overline{g}}\,J_{\overline{g}}(x)dx \\
&= \frac{d}{ds}\Big|_{s=0}\int_{S^{n-1}}\int_{f(\theta)-s}^{f(\theta)}\left|\nabla^{\overline{g}}F(r\theta)\right|_{\overline{g}}\,J_{\overline{g}}(r\theta)r^{n-1}drd\theta \\
&= \int_{S^{n-1}}\sqrt{1+|\nabla^{\overline{g}}f(\theta)|_{\overline{g}}^2}\,J_{\overline{g}}\big(f(\theta)\theta\big)f^{n-1}(\theta)\,d\theta.
\end{aligned}
$$

Now applying Lemma 3.3(iii), we obtain

$$
\mathrm{I}(\Omega_{\overline{g}}) \geqslant \frac{\left(\int_{S^{n-1}}\sqrt{\rho^2+|\nabla^k f|_k^2}\,J_k(f)f^{n-1}d\theta\right)^n}{\left(\int_{S^{n-1}}\rho\,\mathcal{J}_k(f)d\theta\right)^{n-1}}.
$$

So we may write

$$
\mathrm{I}(\Omega_{\overline{g}}) \geqslant \lambda^{n-1}(1), \qquad \text{where} \qquad \lambda(t) := \frac{\left(\int_{S^{n-1}}\sqrt{\rho^{2t}+|\nabla^k f|_k^2}\,J_k(f)f^{n-1}d\theta\right)^{\frac{n}{n-1}}}{\int_{S^{n-1}}\rho^t\mathcal{J}_k(f)d\theta},
$$

for $t\in[0,1]$. Let $A^{\frac{n}{n-1}}$ denote the numerator and B the denominator of $\lambda(t)$. Then

$$
\lambda'(t) = \frac{A^{\frac{1}{n-1}}}{B}\left(\frac{n}{n-1}A' - \frac{A}{B}B'\right) = \frac{A^{\frac{1}{n-1}}}{B}\int_{S^{n-1}}C\ln(\rho)\rho^t J_k(f)f^{n-1}\,d\theta, \tag{4}
$$

where

$$
C := \frac{n}{n-1}\frac{\rho^t}{\sqrt{\rho^{2t}+|\nabla^k f|_k^2}} - \frac{A}{B}\frac{\mathcal{J}_k(f)}{J_k(f)f^{n-1}}.
$$

By Lemma 3.3(ii), $\rho\geqslant 1$. Thus the sign of λ' depends on that of C. By assumption, $|f-R|\leqslant\varepsilon$ and $|\nabla f|\leqslant\varepsilon$. So by (3) and continuity of J_k and $\mathcal{J}_k$, for any $\overline{\varepsilon}>0$ we may choose ε so small that

$$
|f-R|\leqslant\overline{\varepsilon}, \qquad |\nabla^k f|_k\leqslant\overline{\varepsilon}, \qquad \left|\frac{J_k(f)}{J_k(R)}-1\right|\leqslant\overline{\varepsilon}, \quad\text{and}\quad \left|\frac{\mathcal{J}_k(f)}{\mathcal{J}_k(R)}-1\right|\leqslant\overline{\varepsilon}.
$$

Then

$$
\begin{aligned}
A &\leqslant \sqrt{1+\overline{\varepsilon}^2}J_k(R)\,(1+\overline{\varepsilon})\,(R+\overline{\varepsilon})^{n-1}\int_{S^{n-1}}\rho^t d\theta, \\
B &\geqslant \mathcal{J}_k(R)(1-\overline{\varepsilon})\int_{S^{n-1}}\rho^t d\theta.
\end{aligned}
$$

Thus

$$
C \geqslant \frac{n}{n-1}\frac{1}{\sqrt{1+\overline{\varepsilon}^2}} - \left(\frac{1+\overline{\varepsilon}}{1-\overline{\varepsilon}}\right)^2\left(\frac{R+\overline{\varepsilon}}{R-\overline{\varepsilon}}\right)^{n-1}\sqrt{1+\overline{\varepsilon}^2}.
$$

So if $\overline{\varepsilon}$ is sufficiently small, then $C>0$, which yields $\lambda'\geqslant 0$. Thus $\lambda(1)\geqslant\lambda(0)$. But $\lambda^{n-1}(0)=\mathrm{I}(\Omega_{\delta^k})$. So $\mathrm{I}(\Omega_{\overline{g}})\geqslant\mathrm{I}(\Omega_{\delta^k})$ as desired.

Finally suppose that $\mathrm{I}(\Omega_{\overline{g}}) = \mathrm{I}(\Omega_{\delta^k})$. Then $\lambda(1) = \lambda(0)$. So $\lambda' = 0$ identically. Since $C > 0$, it follows from (4) that $\rho = 1$ identically. So $J_{\overline{g}}(f(\theta)\theta) = J_k(f(\theta))$. Consequently, by Lemma 3.3(i), $J_{\overline{g}}(r\theta) \leqslant J_k(r)$ for $r \leqslant f(\theta)$. Thus $\overline{g} = \delta^k$ by Lemma 3.3(ii). □

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References

[1] S. Alexander, V. Kapovitch, and A. Petrunin, *Alexandrov geometry—foundations*, Graduate Studies in Mathematics, vol. 236, American Mathematical Society, Providence, RI, [2024] ©2024. MR4734965 ↑2

[2] S. B. Alexander, I. D. Berg, and R. L. Bishop, *Geometric curvature bounds in Riemannian manifolds with boundary*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 703–716. MR1113693 ↑2

[3] S. B. Alexander and R. L. Bishop, *The Hadamard-Cartan theorem in locally convex metric spaces*, Enseign. Math. (2) **36** (1990), no. 3-4, 309–320. MR1096422 ↑2

[4] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), no. 5, Aii, A279–A281. MR0407905 ↑1

[5] W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics, vol. 61, Birkhäuser Boston, Inc., Boston, MA, 1985. MR823981 ↑1

[6] R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*, Pure and Applied Mathematics, Vol. XV, Academic Press, New York-London, 1964. MR169148 ↑5

[7] A. Borbély, *On the total curvature of convex hypersurfaces in hyperbolic spaces*, Proc. Amer. Math. Soc. **130** (2002), no. 3, 849–854. MR1866041 ↑2

[8] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR1744486 ↑2

[9] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR1835418 (2002e:53053) ↑2

[10] Yu. D. Burago and V. A. Zalgaller, *Geometricheskie neravenstva*, "Nauka" Leningrad. Otdel., Leningrad, 1980. MR602952 ↑1

[11] ———, *Geometric inequalities*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 285, Springer-Verlag, Berlin, 1988. Translated from the 1980 Russian original by A. B. Sosinskiĭ, Springer Series in Soviet Mathematics. MR936419 ↑2, 5

[12] I. Chavel, *Isoperimetric inequalities*, Cambridge Tracts in Mathematics, vol. 145, Cambridge University Press, Cambridge, 2001. Differential geometric and analytic perspectives. MR1849187 ↑1

[13] C. B. Croke, *A sharp four-dimensional isoperimetric inequality*, Comment. Math. Helv. **59** (1984), no. 2, 187–192. MR749103 ↑1, 2

[14] O. Druet, *Sharp local isoperimetric inequalities involving the scalar curvature*, Proc. Amer. Math. Soc. **130** (2002), no. 8, 2351–2361. MR1897460 ↑2

[15] ———, *Isoperimetric inequalities on nonpositively curved spaces*, Lecture Notes for Queen Didot conference, Tunisia (2010), available at `pdfs.semanticscholar.org/ad32/0bbe353be1d5ae528c58012afd9788e59a67.pdf`. ↑2

[16] M. Ghomi and J. Spruck, *Total curvature and the isoperimetric inequality in Cartan-Hadamard manifolds*, J. Geom. Anal. **32** (2022), no. 2, Paper No. 50, 54pp. MR4358702 ↑2

[17] ______, *Minkowski inequality in Cartan-Hadamard manifolds*, Int. Math. Res. Not. IMRN **20** (2023), 17892–17910. MR4659867 ↑2

[18] M. Gromov, *Structures métriques pour les variétés riemanniennes*, Textes Mathématiques [Mathematical Texts], vol. 1, CEDIC, Paris, 1981. Edited by J. Lafontaine and P. Pansu. MR682063 ↑1

[19] D. Hoffman and J. Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Comm. Pure Appl. Math. **27** (1974), 715–727. MR0365424 ↑2

[20] B. Kleiner, *An isoperimetric comparison theorem*, Invent. Math. **108** (1992), no. 1, 37–47. MR1156385 ↑1

[21] B. R. Kloeckner and G. Kuperberg, *The Cartan-Hadamard conjecture and the Little Prince*, Rev. Mat. Iberoam. **35** (2019), no. 4, 1195–1258. MR3988083 ↑2

[22] J. M. Lee, *Introduction to Riemannian manifolds*, Graduate Texts in Mathematics, vol. 176, Springer, Cham, 2018. Second edition of [MR1468735]. MR3887684 ↑5

[23] J. Li, Z. Lin, and L. Xu, *A total curvature estimate of closed hypersurfaces in non-positively curved symmetric spaces*, arXiv preprint, arXiv:2501.17146 (2025), available at `2501.17146`. ↑2

[24] F. Morgan and D. L. Johnson, *Some sharp isoperimetric theorems for Riemannian manifolds*, Indiana Univ. Math. J. **49** (2000), no. 3, 1017–1041. MR1803220 ↑2

[25] S. Nardulli and L. E. Osorio Acevedo, *Sharp isoperimetric inequalities for small volumes in complete noncompact Riemannian manifolds of bounded geometry involving the scalar curvature*, Int. Math. Res. Not. IMRN **15** (2020), 4667–4720. MR4130849 ↑2

[26] R. Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 1182–1238. MR500557 ↑1

[27] M. Ritoré, *Isoperimetric inequalities in Riemannian manifolds*, Progress in Mathematics, vol. 348, Birkhäuser/Springer, Cham, [2023] ©2023. MR4676392 ↑2

[28] A. Weil, *Sur les surfaces a courbure negative*, CR Acad. Sci. Paris **182** (1926), no. 2, 1069–71. ↑1

[29] S. T. Yau, *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. (4) **8** (1975), no. 4, 487–507. MR0397619 ↑2

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