DEFORMATIONS OF CURVES WITH CONSTANT CURVATURE

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Abstract. We prove that curves of constant curvature satisfy the parametric $\mathcal{C}^1$-dense relative h-principle in the space of immersed curves with nonvanishing curvature in Euclidean space $\mathbf{R}^{n\geq 3}$. It follows that two knots of constant curvature in $\mathbf{R}^3$ are isotopic, resp. homotopic, through curves of constant curvature if and only if they are isotopic, resp. homotopic, and their self-linking numbers, resp. self-linking numbers mod 2, are equal.

1. Introduction

The homotopy and isotopy classes of closed curves with nonvanishing curvature in Euclidean space $\mathbf{R}^n$ have been classified in terms of their self-linking number [5, 12, 21]. In this work we obtain parallel results for curves of constant curvature by an approximation argument.

To state our main result, let Γ be an interval $[a, b] \subset \mathbf{R}$ or the topological circle $\mathbf{R}/((b-a)\mathbf{Z})$, and $\mathcal{C}^k(\Gamma, \mathbf{R}^n)$ be the space of $\mathcal{C}^k$ maps $f\colon \Gamma \to \mathbf{R}^n$ with its standard topology induced by the $\mathcal{C}^k$-norm $|\cdot|_k$. We say that f_0, $f_1 \in \mathcal{C}^k(\Gamma, \mathbf{R}^n)$ are *$\mathcal{C}^{\ell\leq k}$-homotopic* if there is a family $f_t \in \mathcal{C}^\ell(\Gamma, \mathbf{R}^n)$, for $t \in [0, 1]$, such that $t \mapsto f_t$ is continuous. Then f_t is a called a *$\mathcal{C}^\ell$-homotopy*. The space of (immersed) *curves* $\mathrm{Imm}^k(\Gamma, \mathbf{R}^n) \subset \mathcal{C}^k(\Gamma, \mathbf{R}^n)$ consists of locally injective maps with nonvanishing derivative when $k \geq 1$. The *curvature* of $f \in \mathrm{Imm}^2(\Gamma, \mathbf{R}^n)$ is given by $\kappa \coloneqq |T'|/|f'|$, where $T \coloneqq f'/|f'|$ is the *tantrix* of f.

Theorem 1.1. *Let $f_t \in \mathrm{Imm}^{k\geq 2}(\Gamma, \mathbf{R}^{n\geq 3})$ be a $\mathcal{C}^2$-homotopy of curves with nonvanishing curvature. Suppose that f_0 and f_1 have constant curvature. Then, for any $\varepsilon > 0$, there exists a $\mathcal{C}^2$-homotopy $\widetilde{f}_t \in \mathrm{Imm}^k(\Gamma, \mathbf{R}^n)$ of curves with constant curvature such that $\widetilde{f}_0 = f_0$, $\widetilde{f}_1 = f_1$, and $|\widetilde{f}_t - f_t|_1 \leq \varepsilon$. Furthermore, if f_t have unit speed, then $\widetilde{f}_t$ may be tangent to f_t at any finite set of points prescribed in Γ.*

In the terminology of Gromov [13] or Eliashberg-Mishachev [4], the result above establishes a parametric $\mathcal{C}^1$-dense relative h-principle for curves of constant curvature in the space of curves with nonvanishing curvature. The nonparametric version, which states that curves of constant curvature are dense in $\mathrm{Imm}^{k\geq 2}(\Gamma, \mathbf{R}^n)$ with respect to the $\mathcal{C}^1$-norm, had been obtained earlier in [8].

We say that $f \in \mathcal{C}^k(\Gamma, \mathbf{R}^n)$ is *closed* when Γ is a circle. If closed curves f_0, $f_1 \in \mathrm{Imm}^{k\geq 2}(\Gamma, \mathbf{R}^3)$ with nonvanishing curvature are $\mathcal{C}^2$-homotopic through curves with nonvanishing curvature, then their tantrices T_0, T_1 are $\mathcal{C}^1$-homotopic. Feldman

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[5] proved the converse of this phenomenon, via Smale's extension [24] of Whitney-Graustein theorem [29] to spherical curves. Furthermore, Rosenberg [21] showed that T_0 and T_1 are $\mathcal{C}^1$-homotopic if and only if $\mathrm{SL}_2(f_0) = \mathrm{SL}_2(f_1)$, where SL_2 denotes the self-linking number mod 2. Thus Theorem 1.1 yields:

Corollary 1.2. *A pair of closed curves f_0, $f_1 \in \mathrm{Imm}^{k\geq 2}(\Gamma, \mathbf{R}^3)$ of constant curvature are $\mathcal{C}^2$-homotopic through $\mathcal{C}^k$ curves of constant curvature if and only if* $\mathrm{SL}_2(f_0) = \mathrm{SL}_2(f_1)$.

Let $\mathrm{Emb}^k(\Gamma, \mathbf{R}^n) \subset \mathrm{Imm}^k(\Gamma, \mathbf{R}^n)$ denote the space of injective or embedded curves. A pair of curves f_0, $f_1 \in \mathrm{Emb}^k(\Gamma, \mathbf{R}^n)$ are $\mathcal{C}^k$-*isotopic* if they can be joined by a $\mathcal{C}^k$-homotopy $f_t \in \mathrm{Emb}^k(\Gamma, \mathbf{R}^n)$. A *knot* is an embedded closed curve $f \in \mathrm{Emb}^0(\Gamma, \mathbf{R}^3)$. Gluck and Pan [12] showed that two knots of nonvanishing curvature f_0, $f_1 \in \mathrm{Emb}^{k\geq 2}(\Gamma, \mathbf{R}^n)$ are $\mathcal{C}^2$-isotopic through knots of nonvanishing curvature if and only if they are $\mathcal{C}^0$-isotopic and $\mathrm{SL}(f_0) = \mathrm{SL}(f_1)$, i.e., they have the same self-linking number. If f_t in Theorem 1.1 is an isotopy, then (choosing ε sufficiently small) we may assume that $\widetilde{f}_t$ is an isotopy as well. Thus we obtain:

Corollary 1.3. *A pair of knots f_0, $f_1 \in \mathrm{Emb}^{k\geq 2}(\Gamma, \mathbf{R}^3)$ of constant curvature are $\mathcal{C}^2$-isotopic through $\mathcal{C}^k$ knots of constant curvature if and only if they are $\mathcal{C}^0$-isotopic and* $\mathrm{SL}(f_0) = \mathrm{SL}(f_1)$.

So the $\mathcal{C}^2$ homotopy and isotopy classes of closed curves of constant curvature in $\mathbf{R}^3$ mirror those of closed curves with nonvanishing curvature. In other words, a loop of constant curvature is just as flexible as one without inflection points. The self-linking number of a knot $f \in \mathrm{Emb}^2(\Gamma, \mathbf{R}^3)$ with nonvanishing curvature is the linking number of f with a perturbation of f along its principal normal $N := T/\kappa$. The self-linking number mod 2 of a closed curve $f \in \mathrm{Imm}^2(\Gamma, \mathbf{R}^3)$ with nonvanishing curvature is the self-linking number mod 2 of a perturbation of f to an embedded curve. See [12, 20, 21] for more background and references on self-linking number.

The proof of Theorem 1.1 utilizes the fact that, when $|f_t'| = 1$, the curvature of f_t is the speed of its tantrix T_t. Thus we approximate T_t with a homotopy $\widetilde{T}_t$ of curves with constant speed, and set $\widetilde{f}_t(s) := \int_a^s \widetilde{T}_t(u)du$. For $\widetilde{f}_t$ to be closed when f_t is closed, we need to have $\int_\Gamma \widetilde{T}_t = \int_\Gamma T_t$. To this end, we construct $\widetilde{T}_t$ by creating sinusoidal bumps near suitably chosen points along T_t, via parametric versions of classical theorems of Carathéodory and Steinitz. The bumps have to be long enough to control $\int_\Gamma \widetilde{T}_t$, while vanishing sufficiently fast as $t \to 0$, 1 to ensure that $\widetilde{T}_t$ is a $\mathcal{C}^1$-homotopy. Implementing this plan is particularly subtle when the convex hull of T_0 or T_1 has no interior points, as we will discuss below.

Constructing closed curves of constant curvature, by integrating spherical curves, goes back to Fenchel [6], and belongs to the convex integration theory [4, 7, 13, 25], which includes works by Whitney [29] and Nash-Kuiper [17, 19]. A nonparametric version of the above argument had been developed in [8]; however, the perturbation method there is not suitable for obtaining a $\mathcal{C}^1$-homotopy of tantrices, and our present construction is fundamentally different. See also [11] for another nonparametric approach, and [1,10] for some other examples of parametric h-principle in Riemannian geometry.

Nontrivial $\mathcal{C}^2$ knots of constant curvature were first constructed by Koch and Engelhardt [16] by gluing helical segments. Using the same method, McAtee [18] obtained $\mathcal{C}^2$ knots in every isotopy class. Existence of $\mathcal{C}^\infty$ knots of constant curvature, in every isotopy class, was first established in [8] via convex integration. This approach was also used later by Wasem [27] to obtain an h-principle for $\mathcal{C}^2$ curves with prescribed curvature.

2. Reduction to a Local Problem

We start by showing that Theorem 1.1 is a consequence of the following result which supplies additional information on the curvature of the desired homotopy $\widetilde{f}_t$ and can be applied to subintervals in any partition of Γ. Let $I := [a, b]$. It is always assumed that $t \in [0, 1]$ unless specified otherwise.

Theorem 2.1. *Let $f_t \in \mathrm{Imm}^{k\geq 2}(I, \mathbf{R}^{n\geq 3})$ be a $\mathcal{C}^2$-homotopy of unit speed curves with nonvanishing curvature κ_t. Suppose that κ_0 and κ_1 are constant. Then for any $\varepsilon > 0$ there exists a $\mathcal{C}^2$-homotopy $\widetilde{f}_t \in \mathrm{Imm}^k(I, \mathbf{R}^n)$ of unit speed curves with constant curvature $\widetilde{\kappa}_t$ such that*

1. *$\widetilde{f}_0 = f_0$ and $\widetilde{f}_1 = f_1$,*
2. *$|\widetilde{f}_t - f_t|_1 \leq \varepsilon$,*
3. *$\widetilde{f}_t = f_t$ on ∂I,*
4. *$\widetilde{T}_t(\widetilde{U}_t) = T_t(U_t)$ for some neighborhoods $\widetilde{U}_t$, U_t of ∂I.*

Furthermore, $\widetilde{\kappa}_t$ may be set equal to any continuous function $c(t)$ such that $c(0) = \kappa_0$, $c(1) = \kappa_1$, $c(t) > \max(\kappa_t)$ for $t \in (0, 1)$, and $c(t)$ is sufficiently large on $[\eta, 1 - \eta]$ where $\eta > 0$ is sufficiently small.

The last statement in the above theorem means that there exists $\overline{\eta} > 0$ such that any $0 < \eta \leq \overline{\eta}$ is admissible; and that for every such η there exists a number $\overline{c} > 0$ such that any function $c(t) \geq \overline{c}$ on $[\eta, 1 - \eta]$ is admissible.

Suppose that Theorem 2.1 holds. Then Theorem 1.1 is established as follows. We may identify Γ with I by noting that $f_t \in \mathrm{Imm}^k(I, \mathbf{R}^n)$ is closed provided that $f_t(a) = f_t(b)$ and the derivatives $f_t^{(i)}(a) = f_t^{(i)}(b)$ for $i \leq k$. We may also assume that f_t in Theorem 1.1 has unit speed. Indeed after replacing f_t with the rescalings $|I| f_t / \operatorname{length}(f_t)$, where $|I| = b - a$, we may assume that f_t have constant length $|I|$. Then there exists a continuous family of diffeomorphisms $\theta_t \colon I \to I$ such that $f_t \circ \theta_t$ has unit speed. Now if we obtain a homotopy $\widetilde{f}_t$ with $|\widetilde{f}_t - f_t \circ \theta_t|_1$ sufficiently small, then it is easily seen that $|\widetilde{f}_t \circ \theta_t^{-1} - f_t|_1$ will also be small; see the proof of [8, Prop. 5.4]. Hence $\widetilde{f}_t \circ \theta_t^{-1}$ will be the desired homotopy.

To find $\widetilde{f}_t$, let $a =: s_0 < \cdots < s_N := b$ be any partition of I which includes the points prescribed in Γ, and set $I_i := [s_i, s_{i+1}]$. Let f_t^i be the restriction of f_t to I_i, and $c_i(t)$ be an admissible curvature function for deforming f_t^i as described in Theorem 2.1. Then $c(t) := \sum_i c_i(t)$ will also be admissible. Applying Theorem 2.1 to f_t^i with respect to $c(t)$ and ε, we obtain homotopies $\widetilde{f}_t^i$ such that $|(\widetilde{f}_t^i)'| = 1$,

$|\widetilde{f}_t^i - f_t^i|_1 \leq \varepsilon$, and $\widetilde{f}_t^i$ have uniform constant curvature. Set $\widetilde{f}_t := \widetilde{f}_t^i$ on I_i. We only need to check that $\widetilde{f}_t$ is $\mathcal{C}^k$. Then it follows immediately that $|\widetilde{f}_t - f_t|_1 \leq \varepsilon$ and $\widetilde{f}_t$ has constant curvature as desired.

To see that $\widetilde{f}_t$ is $\mathcal{C}^k$, first note that by item (3) in Theorem 2.1, $\widetilde{f}_t^i = f_t^i = f_t$ on ∂I_i. Thus $\widetilde{f}_t$ is $\mathcal{C}^0$. Furthermore, since $|(\widetilde{f}_t^i)'| = 1$, $\widetilde{T}_t^i = (\widetilde{f}_t^i)'$. So by item (4) in Theorem 2.1, $(\widetilde{f}_t^i)' = (f_t^i)' = f_t'$ on ∂I_i, which shows that $\widetilde{f}_t$ is $\mathcal{C}^1$. In particular $\widetilde{T}_t = \widetilde{f}_t'$ is well-defined. So it suffices to check that $\widetilde{T}_t$ is $\mathcal{C}^{k-1}$. Note that $\widetilde{T}_t$ is piecewise $\mathcal{C}^{k-1}$ since $\widetilde{T}_t|_{I_i} = \widetilde{T}_t^i = (\widetilde{f}_t^i)'$ and $\widetilde{f}_t^i$ is $\mathcal{C}^k$ by Theorem 2.1. Furthermore, by item (4) of Theorem 2.1, if s_i lies in the interior of I, then there are neighborhoods U_{i-1} and U_i of s_i in I_{i-1} and I_i respectively such that $\widetilde{T}_t(U_{i-1}) = T_t(U_{i-1})$ and $\widetilde{T}_t(U_i) = T_t(U_i)$. Hence $\widetilde{T}_t$ is reparametrization of T_t with constant speed on the neighborhood $U_{i-1} \cup U_i$ of $s_i \in I$. Since T_t is $\mathcal{C}^{k-1}$, it follows that $\widetilde{T}_t$ is $\mathcal{C}^{k-1}$ near s_i. So $\widetilde{T}_t$ is $\mathcal{C}^{k-1}$ as claimed.

3. Approximation with Nonflat Curves

Here we show that homotopies $f_t \in \mathrm{Imm}^k(I, \mathbf{R}^n)$ may be perturbed to assume a generic property, which we need in order to establish Theorem 2.1. First we record the following fact, which is a standard application of Thom's jet transversality theorem [4, 14].

Lemma 3.1. *Let $f \in \mathcal{C}^{k\geq 1}(I, \mathbf{R}^n)$. Then for any $\varepsilon > 0$ there exists $\overline{f} \in \mathrm{Imm}^\infty(I, \mathbf{R}^n)$ such that $|\overline{f} - f|_k \leq \varepsilon$ and the first n derivatives of $\overline{f}$ are linearly independent except possibly at finitely many points of I.*

Proof. Let $J^n(I, \mathbf{R}^n) = I \times \mathbf{R}^{n(n+1)}$ be the space of the n-jets of maps $f \in \mathcal{C}^n(I, \mathbf{R}^n)$. The n-jet $j^n f \colon I \to J^n(I, \mathbf{R}^n)$ is given by $j^n f(s) = (s, f(s), f'(s), \ldots, f^{(n)}(s))$. Let $A \subset J^n(I, \mathbf{R}^n)$ consist of elements whose last n-coordinates are linearly dependent. The set of last n elements of A may be identified with $n \times n$ matrices of rank less than n, which form a stratified space of dimension $n^2 - 1$ [4, p. 16]. So $\dim(A) = 1 + n + (n^2 - 1)$ or $\mathrm{codim}(A) = 1$. By the jet transversality theorem, for an open dense set of maps $\overline{f} \in \mathcal{C}^n(I, \mathbf{R}^n)$, $j^n \overline{f}$ is transversal to A. Since $\dim(I) = \mathrm{codim}(A)$, $j^n \overline{f}$ may intersect A only at a discrete set. Finally, since $\mathrm{Imm}^\infty(I, \mathbf{R}^n)$ is dense in $\mathcal{C}^n(I, \mathbf{R}^n)$ [14, Thm. 2.12], we may assume that $\overline{f} \in \mathrm{Imm}^\infty(I, \mathbf{R}^n)$. □

A subset of $\mathbf{R}^n$ is *nonflat* if its convex hull has interior points. By abuse of notation, for any mapping $f \in \mathcal{C}^k(I, \mathbf{R}^n)$, we may write f to refer to $f(I)$, when the meaning is clear from context, e.g., we say f is nonflat, if $f(I)$ is nonflat. Note that if derivatives of $f \in \mathcal{C}^n(I, \mathbf{R}^n)$ are linearly independent up to order n, at some point $s \in I$, then f is nonflat.

Proposition 3.2. *Let $f_t \in \mathrm{Imm}^{k\geq 2}(I, \mathbf{R}^{n\geq 3})$ be a $\mathcal{C}^k$-homotopy with nonvanishing curvature. Then, for any $\varepsilon > 0$, there exists a $\mathcal{C}^k$ homotopy $\overline{f}_t \in \mathrm{Imm}^k(I, \mathbf{R}^n)$ with nonvanishing curvature such that $\overline{f}_0 = f_0$, $\overline{f}_1 = f_1$, $|f_t - \overline{f}_t|_k \leq \varepsilon$, $\overline{f}_t$ is nonflat for $t \in (0, 1)$, and $\overline{f}_t = f_t$ on a neighborhood of ∂I.*

Proof. For $i \in \mathbf{Z}$ choose $t_i \in (0,1)$ with $t_i < t_{i+1}$ and $t_i \to 1$ or 0 as $i \to \infty$ or $-\infty$ respectively. By Lemma 3.1, there are curves $g_{t_i} \in \mathrm{Imm}^\infty(I, \mathbf{R}^n)$ such that every segment of g_{t_i} is nonflat, and $|g_{t_i} - f_{t_i}|_k \to 0$ as $i \to \pm\infty$. For $\lambda \in [0,1]$, set

$$\overline{f}_{(1-\lambda)t_i + \lambda t_{i+1}} := (1-\lambda) g_{t_i} + \lambda g_{t_{i+1}}. \tag{1}$$

This defines a homotopy $\overline{f}_t \in \mathcal{C}^\infty(I, \mathbf{R}^n)$ for $t \in (0,1)$ with $|\overline{f}_t - f_t|_k \to 0$ as $t \to 1, 0$. So setting $\overline{f}_0 := f_0$ and $\overline{f}_1 := f_1$ yields a $\mathcal{C}^k$-homotopy $\overline{f}_t \in \mathcal{C}^k(I, \mathbf{R}^n)$ for $t \in [0,1]$.

Since $t \mapsto |f_t|_k$ is continuous, by Lemma 3.1 we may choose t_i and g_{t_i} so that

$$|f_t - f_{t'}|_k \leq \delta, \text{ for } t, t' \in [t_i, t_{i+1}], \qquad \text{and} \qquad |g_{t_i} - f_{t_i}|_k \leq \delta, \tag{2}$$

for any given $\delta > 0$. For $t \in [t_i, t_{i+1}]$ we may rewrite (1) as $\overline{f}_t = (1-\lambda_t) g_{t_i} + \lambda_t g_{t_{i+1}}$, where $\lambda_t \in [0,1]$ is the number such that $t = (1-\lambda_t)t_i + \lambda_t t_{i+1}$. Then,

$$\begin{aligned} |\overline{f}_t - f_t|_k &\leq (1-\lambda_t)|g_{t_i} - f_t|_k + \lambda_t |g_{t_{i+1}} - f_t|_k \\ &\leq |g_{t_i} - f_{t_i}|_k + |f_{t_i} - f_t|_k + |g_{t_{i+1}} - f_{t_{i+1}}|_k + |f_{t_{i+1}} - f_t|_k \leq 4\delta. \end{aligned} \tag{3}$$

So setting $\delta \leq \varepsilon/4$, we obtain $|\overline{f}_t - f_t|_k \leq \varepsilon$ as desired. Then choosing ε sufficiently small yields that $\overline{f}_t \in \mathrm{Imm}^k(I, \mathbf{R}^n)$ and $\overline{f}_t$ has nonvanishing curvature.

To make $\overline{f}_t$ nonflat, let $\phi \colon I \to [0,1]$ be a $\mathcal{C}^\infty$ step function with $\phi = 0$ near 0 and $\phi = 1$ near 1. Set $\bar{t}_i := (t_i + t_{i+1})/2$, and $g_{\bar{t}_i} := \phi g_{t_i} + (1-\phi) g_{t_{i+1}}$. Then $g_{\bar{t}_i}$ is nonflat, since it shares a segment with g_{t_i}. We have

$$|g_{\bar{t}_i} - f_{\bar{t}_i}|_k = |\phi(g_{t_i} - g_{t_{i+1}}) + g_{t_{i+1}} - f_{\bar{t}_i}|_k \leq C|g_{t_i} - g_{t_{i+1}}|_k + |g_{t_{i+1}} - f_{\bar{t}_i}|_k,$$

where C is a constant which depends only on $|\phi|_k$. Furthermore,

$$|g_{t_i} - g_{t_{i+1}}|_k \leq |g_{t_i} - f_{t_i}|_k + |f_{t_i} - f_{t_{i+1}}|_k + |f_{t_{i+1}} - g_{t_{i+1}}|_k \leq 3\delta,$$
$$|g_{t_{i+1}} - f_{\bar{t}_i}|_k \leq |g_{t_{i+1}} - f_{t_{i+1}}|_k + |f_{t_{i+1}} - f_{\bar{t}_i}|_k \leq 2\delta.$$

So $|g_{\bar{t}_i} - f_{\bar{t}_i}|_k \leq (3C+2)\delta$. Set $\delta \leq \varepsilon/(4(3C+2))$, replace $\{t_i\}$ by a reindexing of $\{t_i, \bar{t}_i\}$, and let $\overline{f}_t$ be the corresponding homotopy given by (1). Then $|g_{t_i} - f_{t_i}|_k \leq \varepsilon/4$ by (2), which yields $|\overline{f}_t - f_t|_k \leq \varepsilon$ by (3). Furthermore, $\overline{f}_t$ is now nonflat, since it shares a segment with g_{t_i} or $g_{t_{i+1}}$, when $t \in [t_i, t_{i+1}]$. Finally, using a partition of unity on I, we may glue $\overline{f}_t$ to f_t near ∂I. ☐

4. The Length and Center of Mass of Spherical Curves

Here we reduce Theorem 2.1 to a geometric result for nonflat spherical curves, which is applied to the tantrix of f_t. The *center of mass* and *average* of a curve $f \in \mathrm{Imm}^1(I, \mathbf{R}^n)$ are defined as

$$\mathrm{cm}(f) := \frac{1}{\mathrm{length}(f)} \int_I f|f'|, \qquad \text{and} \qquad \mathrm{ave}(f) := \frac{1}{|I|} \int_I f.$$

Note that $\mathrm{cm}(f)$ is invariant under reparametrizations of f, and when $|f'|$ is constant, $\mathrm{cm}(f) = \mathrm{ave}(f)$. Let $\mathrm{conv}(f)$ denote the convex hull of f. If f is nonflat, then $\mathrm{cm}(f)$ and $\mathrm{ave}(f)$ lie in the interior of $\mathrm{conv}(f)$ [8, Lem. 2.3] which is denoted by $\mathrm{int}\,\mathrm{conv}(f)$. We say $f \in \mathcal{C}^k(I, \mathbf{S}^{n-1})$ is nonflat if f is nonflat in $\mathbf{R}^n$. Let $U_\varepsilon(f)$ denote the open neighborhood of f of radius ε.

Theorem 4.1. *Let* $f_t \in \mathrm{Imm}^{k\geq 1}(I, \mathbf{S}^{n-1})$ *be a* $\mathcal{C}^1$*-isotopy of nonflat curves of length* ℓ_t*, and* $x\colon [0,1] \to \mathbf{R}^n$ *be a continuous map with* $x(t) \in \mathrm{int}\,\mathrm{conv}(f_t)$*. Suppose that* $x(0) = \mathrm{cm}(f_0)$ *and* $x(1) = \mathrm{cm}(f_1)$*. Then for any* $\varepsilon > 0$ *and natural number* m*, there exists a* $\mathcal{C}^1$*-isotopy* $\widetilde{f}_t \in \mathrm{Imm}^k(I, \mathbf{S}^{n+m-1})$ *such that*

(1) $\widetilde{f}_0 = f_0$ *and* $\widetilde{f}_1 = f_1$*,*
(2) $\widetilde{f}_t \subset U_\varepsilon(f_t)$*,*
(3) $\mathrm{cm}(\widetilde{f}_t) = x(t)$*,*
(4) $\widetilde{f}_t = f_t$ *on a neighborhood of* ∂I*,*
(5) $\widetilde{f}_t$ *is nonflat in* $\mathbf{R}^{n+m}$ *for* $t \in (0,1)$*.*

Furthermore, we may set $\mathrm{length}(\widetilde{f}_t)$ *equal to any continuous function* $\widetilde{\ell}_t$ *such that* $\widetilde{\ell}_0 = \ell_0$*,* $\widetilde{\ell}_1 = \ell_1$*,* $\widetilde{\ell}_t > \ell_t$ *for* $t \in (0,1)$*, and* $\widetilde{\ell}_t$ *is sufficiently large on* $[\eta, 1-\eta]$ *where* $\eta > 0$ *is sufficiently small.*

The last statement in the above theorem means that there exists $\overline{\eta} > 0$ such that any $0 < \eta \leq \overline{\eta}$ is admissible; and that for every such η there exists a number $\overline{\ell} > 0$ such that any function $\widetilde{\ell}_t \geq \overline{\ell}$ on $[\eta, 1-\eta]$ is admissible.

To show that Theorem 2.1 follows from Theorem 4.1, first we record:

Corollary 4.2. *Let* $f_0 \in \mathrm{Imm}^{k\geq 2}(I, \mathbf{R}^n)$ *be a curve with constant positive curvature, and* I_i *be subintervals in a partition of* I*. Then there exists a* $\mathcal{C}^2$*-homotopy* $\widetilde{f}_t \in \mathrm{Imm}^k(I, \mathbf{R}^n)$ *with constant curvature such that* $\widetilde{f}_0 = f_0$*,* $\widetilde{f}_t = f_0$ *on* ∂I*,* $\widetilde{T}_t(\widetilde{U}_t) = T_0(U_0)$ *for some neighborhoods* $\widetilde{U}_t$*,* U_0 *of* ∂I*, and* $\widetilde{f}_t$ *is nonflat on* I_i *for* $t \in (0,1)$*.*

Proof. We may assume that $|f_0'| = 1$. Then $|T_0'|$ is the curvature of f_0. So T_0 has constant speed. Let $T_t := T_0$, T_t^i be the restriction of T_t to I_i, and set $x_i(t) := \mathrm{ave}(T_0^i)$. Let $c\colon [0,1] \to \mathbf{R}$ be a continuous function such that $c(0) = 1$, $c(t) \geq 1$, and $c(t)$ is so large outside a small neighborhoods of 0, 1 that $\widetilde{\ell}_t^i := c(t)|I_i|$ forms admissible length functions for deforming T_t^i as described in Theorem 4.1. Note that if f_0 is flat, then T_0 lies in a subsphere of $\mathbf{S}^{n-1}$, i.e., the intersection of $\mathbf{S}^{n-1}$ with a subspace of $\mathbf{R}^n$, where it is nonflat. Thus we may apply Theorem 4.1 to T_t^i with respect to $x_i(t)$ and $\widetilde{\ell}_t^i$ to obtain a homotopy $\widetilde{T}_t^i \in \mathrm{Imm}^k(I, \mathbf{S}^{n-1})$ such that $\widetilde{T}_t^i$ is nonflat in $\mathbf{R}^n$ for $t \in (0,1)$, $\widetilde{T}_t^i = T_0$ on a neighborhood of ∂I_i, $\mathrm{cm}(\widetilde{T}_t^i) = \mathrm{ave}(T_0^i)$, and $\mathrm{length}(\widetilde{T}_t^i) = c(t)|I_i|$. The last condition allows us to parametrize $\widetilde{T}_t^i$ with constant speed $c(t)$. Define $\widetilde{T}_t$ by setting it equal to $\widetilde{T}_t^i$ on each I_i. Then $\widetilde{T}_t(\widetilde{U}_t) = T_0(U_0)$ for some open neighborhoods $\widetilde{U}_t$ and U_0 of ∂I, because we had $\widetilde{T}_t^i = T_0$ on a neighborhood of ∂I_i before we reparametrized $\widetilde{T}_t^i$ with constant speed. Now set $\widetilde{f}_t(s) := f_0(a) + \int_a^s \widetilde{T}_t$. Since $\widetilde{T}_t$ is nonflat on I_i for $t \in (0,1)$, so is $\widetilde{f}_t$. Furthermore, since $\widetilde{T}_t^i$ has constant speed $\mathrm{ave}(\widetilde{T}_t^i) = \mathrm{cm}(\widetilde{T}_t^i) = \mathrm{ave}(T_0^i)$. So $\int_{I_i} \widetilde{T}_t = \int_{I_i} T_0$, which yields that $\int_I \widetilde{T}_t = \int_I T_0$. Thus $\widetilde{f}_t = f_0$ on ∂I, which completes the proof. □

The last observation shows that in Theorem 2.1 we may assume that f_0 and f_1 are nonflat on any given partition I_i of I. Indeed, by Corollary 4.2, f_0 and f_1 are $\mathcal{C}^2$-homotopic through curves of constant curvature to a pair of curves, say

$\overline{f}_0$ and $\overline{f}_1$, which are nonflat on I_i. Concatenating these homotopies with f_t, we obtain a $\mathcal{C}^2$-homotopy between $\overline{f}_0$ and $\overline{f}_1$ through curves of nonvanishing curvature, which after a perturbation we may assume to be nonflat by Proposition 3.2. Now if Theorem 2.1 holds when f_0 and f_1 are nonflat on I_i, then $\overline{f}_0$ and $\overline{f}_1$ may be joined by a $\mathcal{C}^2$-homotopy of curves of constant curvature. Concatenating this homotopy with the homotopies between f_0, $\overline{f}_0$ and f_1, $\overline{f}_1$, we obtain a $\mathcal{C}^2$-homotopy between f_0 and f_1 through curves of constant curvature, as desired.

Suppose now that Theorem 4.1 holds. Then we prove Theorem 2.1 as follows. Since f_t in Theorem 2.1 has unit speed, $T_t = f_t'$. So to have $|f_t - \widetilde{f}_t|_1 \leq \varepsilon$ we must have $|T_t - \widetilde{T}_t|_0 \leq \varepsilon$. After subdividing I into finitely many subintervals I_i we may assume that $\mathrm{length}(T_t(I_i)) < \varepsilon$. Then $T_t(I_i)$ is contained in a ball B of radius ε centered at its midpoint. Now if $\widetilde{T}_t(I_i) \subset U_{\varepsilon/2}(T_i(I_i))$, then $\widetilde{T}_t(I_i)$ is contained in B as well. So $|\widetilde{T}_t - T_t|_0 \leq \varepsilon$ on I_i. Furthermore, by Proposition 3.2, we may assume that f_t is nonflat on each I_i for $t \in (0,1)$, which yields that so is T_t. As discussed above, we may also assume that f_0 and f_1 are nonflat on I_i via Corollary 4.2. Then so are T_0 and T_1. Hence we may assume that T_t is nonflat on I_i.

Similar to the proof of Corollary 4.2, let T_t^i be the restriction of T_t to I_i. We may apply Theorem 4.1 to T_t^i with respect to $x_i(t) := \mathrm{ave}(T_t^i)$ and $\widetilde{\ell}_t^i = c(t)|I_i|$ to obtain the homotopy $\widetilde{T}_t^i$ with $\mathrm{cm}(\widetilde{T}_t^i) = \mathrm{ave}(T_t^i)$ and $\mathrm{length}(\widetilde{T}_t^i) = c(t)|I_i|$. So we may assume that $\widetilde{T}_i$ has constant speed $c(t)$, and define $\widetilde{T}_t$ on I by setting it equal to $\widetilde{T}_t^i$ on I_i. Next we set $\widetilde{f}_t(s) := f_t(a) + \int_a^s \widetilde{T}_t$. Then $|\widetilde{f}_t'| = |\widetilde{T}_t| = 1$. So $\widetilde{\kappa}_t = |\widetilde{T}_t'| = c(t)$ as desired. Furthermore, since $\widetilde{T}_t^i$ has constant speed, $\mathrm{ave}(\widetilde{T}_t^i) = \mathrm{cm}(\widetilde{T}_t^i) = \mathrm{ave}(T_t^i)$. So $\int_{I_i} \widetilde{T}_t = \int_{I_i} T_t$, which yields that $\int_I \widetilde{T}_t = \int_I T_t$. Thus $\widetilde{f}_t = f_t$ on ∂I. Next note that $|\widetilde{f}_t(s) - f_t(s)| \leq \int_a^s |\widetilde{T}_t - T_t| \leq \int_I |\widetilde{T}_t - T_t| = \sum_{i=1}^N \int_{I_i} |\widetilde{T}_t - T_t| \leq N\varepsilon|I|$. Thus $|f_t - \widetilde{f}_t|_0 \leq N\varepsilon|I|$. We also have $|f_t' - \widetilde{f}_t'|_0 = |T_t(u) - \widetilde{T}_t(u)|_0 \leq \varepsilon$. Hence $|f_t - \widetilde{f}_t|_1$ can be made arbitrarily small.

5. Perturbation of Geodesics

Here we prove a very special case of Theorem 4.1 by an explicit elementary argument. A curve $f \in \mathrm{Emb}^\infty(I, \mathbf{S}^{n-1})$ is a *geodesic* if it has constant speed and lies in a plane through the origin of $\mathbf{R}^n$.

Lemma 5.1. *For $t \in (0,1]$, let $f_t \in \mathrm{Emb}^\infty(I, \mathbf{S}^{n-1})$ be a $\mathcal{C}^1$-isotopy of geodesics of length ℓ_t. Then, for any continuous family of constants $\widetilde{\ell}_t > \ell_t$ and $\varepsilon > 0$, there exists a $\mathcal{C}^1$-isotopy $\widetilde{f}_t \in \mathrm{Emb}^\infty(I, \mathbf{S}^{n-1})$ such that $\widetilde{f}_t \subset U_\varepsilon(f_t)$, $\widetilde{f}_t = f_t$ on a neighborhood of ∂I, $\mathrm{length}(\widetilde{f}_t) = \widetilde{\ell}_t$, $\mathrm{cm}(\widetilde{f}_t) = \mathrm{cm}(f_t)$, and $\widetilde{f}_t$ is nonflat. Furthermore if $\widetilde{\ell}_t/\ell_t \to 1$ as $t \to 0$, then $|\widetilde{f}_t - f_t|_1 \to 0$.*

Proof. We may assume that $I = [-1, 1]$. After composing f_t with a family of rotations we may also assume that f_t lies in the plane of the first two coordinates, and $f_t(0) = p := (1, 0, \dots, 0)$. So $\mathrm{cm}(f_t) = |\mathrm{cm}(f_t)|p$. Let e_n be the standard basis of $\mathbf{R}^n$. We construct $\widetilde{f}_t$ through a sequence of $n-2$ perturbations of f_t in the directions

$e_3, \dots, e_n$ which increase the length by $(\widetilde{\ell}_t - \ell_t)/(n-2)$ at each step, and preserve $\mathrm{cm}(f_t)$. This is achieved using the $\mathcal{C}^\infty$ bump function $\beta_t \colon \mathbf{R} \to \mathbf{R}$ given by

$$\beta_t(s) := \lambda_t \exp\left(\frac{-1}{\rho_t^2 - s^2}\right)\sin(\omega_t s),$$

if $-\rho_t \leq s \leq \rho_t$, and 0 otherwise. Set $\lambda_t := \min\{|\widetilde{\ell}_t - \ell_t|, \varepsilon/4\}$, and $\rho_t := \min\{m_t, 1 - m_t\}$ for $m_t \in (0,1)$. So β_t depends only on m_t and ω_t, which control its location and length respectively. For $s \in [0,1]$ and appropriate values of m_t and ω_t specified below, we define

$$f_t^1(s) := \frac{f_t(s) + \beta_t(s - m_t)e_3}{|f_t(s) + \beta_t(s - m_t)e_3|},$$

if $m_t - \rho_t \leq s \leq m_t + \rho_t$, and we set $f_t^1(s) := f_t(s)$ otherwise. Then we extend f_t^1 to $[-1,1]$ by letting $f_t^1(-s)$ be the reflection $f_t^1(s)$ in $\mathbf{S}^{n-1}$ with respect to p. See Figure 1. Since f_t^1 is symmetric with respect to p, $\mathrm{cm}(f_t^1) = |\mathrm{cm}(f_t^1)|p$. Furthermore

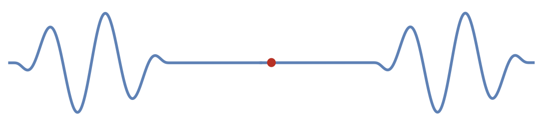

FIGURE 1.

there exists a unique $\omega_t \geq 0$ for each m_t such that $\mathrm{length}(f_t^1) = \ell_t + (\widetilde{\ell}_t - \ell_t)/(n-2)$, since the length of (the graph of) β_t increases monotonically with ω_t. Finally note that if m_t is close to 0, then the bumps are close to p, so $|\mathrm{cm}(f_t^1)| \geq |\mathrm{cm}(f_t)|$ and when m_t is close to 1 then the bumps are away from p so $|\mathrm{cm}(f_t^1)| \leq |\mathrm{cm}(f_t)|$. It follows that there exists a unique choice for m_t such that $|\mathrm{cm}(f_t^1)| = |\mathrm{cm}(f_t)|$. So we have specified a unique choice for f_t^1. Since the parameters ℓ_t, $\widetilde{\ell}_t$, λ_t, ω_t, m_t, and ρ_t all depend continuously on t, $t \mapsto |f_t^1|_1$ is continuous. Similarly, this procedure can be continued, for $2 \leq i \leq n-2$, by setting, for $s \in [0,1]$,

$$f_t^i(s) := \frac{f_t^{i-1}(s) + \beta_t(s - m_t)e_{i+2}}{|\, f_t^{i-1}(s) + \beta_t(s - m_t)e_{i+2}|},$$

if $m_t - \rho_t \leq s \leq m_t + \rho_t$, and $f_t^i(s) := f_t^{i-1}(s)$ otherwise, with appropriate values of m_t and ω_t at each step, as described above. Then $\widetilde{f}_t := f_t^{n-2}$ is the desired homotopy. Finally suppose that $\widetilde{\ell}_t/\ell_t \to 1$. We want to show that then $|\widetilde{f}_t - f_t|_1 \to 0$. To this end note that after composing f_t and $\widetilde{f}_t$ with local homotheties in $\mathbf{S}^{n-1}$ centered at p, we may assume that ℓ_t is constant. Consequently, since $|\widetilde{\ell}_t - \ell_t| \to 0$, both the amplitude λ_t and frequency ω_t of the bumps vanish at each step, while the width of the bumps ρ_t is bounded below by a positive constant. Hence $|\widetilde{f}_t - f_t|_1 \to 0$ as desired. $\square$

6. Parametric Carathéodory and Steinitz Theorems

Here we prove one-parameter versions of some classical results in convex geometry needed to establish Theorem 4.1. Steinitz [26] [22, Thm. 1.3.10] showed that any point in the interior of the convex hull of a set $X \subset \mathbf{R}^n$ lies in the interior of the convex hull of at most $2n$ points of X. We need the following analogue of that result. We say that a family of sets $S_t = \{s_1(t), \dots, s_N(t)\} \subset I$ is continuous if $s_i(t)$ is continuous.

Lemma 6.1. *Let $f_t \in \mathrm{Emb}^0(I, \mathbf{R}^{n\geq 3})$ be a $\mathcal{C}^0$-homotopy, and $x\colon [0,1] \to \mathbf{R}^n$ be a continuous map with $x(t) \in \operatorname{int}\operatorname{conv}(f_t)$. Then there exists a continuous family $S_t \subset I$ of sets of $4n$ distinct points such that $x(t) \in \operatorname{int}\operatorname{conv}(f_t(S_t))$.*

Proof. By Steinitz theorem, for each $t \in [0,1]$ there is a collection $S_t \subset I$ of $2n$ distinct points such that $x(t) \in \operatorname{int}\operatorname{conv}(f_t(S_{t_0}))$. By continuity of $t \mapsto f_t$, for each $t_0 \in [0,1]$ we have $x(t) \in \operatorname{int}\operatorname{conv}(f_t(S_{t_0}))$ for t close to t_0. By Lebesgue number lemma, there exists a partition $0 = t_0 < \cdots < t_\ell = 1$ of $[0,1]$ such that $x(t) \in \operatorname{int}\operatorname{conv}(f_t(S_{t_i}))$ for $t \in [t_i, t_{i+1}]$. Let S_t^e be a continuous selection of $2n$ points such that $S_t^e = S_{t_i}$ whenever $t \in [t_i, t_{i+1}]$ and i is even. Similarly let S_t^o be a continuous selections of $2n$ points such that $S_t^o = S_{t_i}$ whenever $t \in [t_i, t_{i+1}]$ and i is odd. Then $S_t := S^e(t) \cup S^o(t)$ gives a continuous selection of $4n$ points which contain $x(t)$ within the interior of their convex hull; however, S_t may contain fewer than $4n$ points.

To complete the proof it remains to show that there is a continuous selection of $4n$ distinct points of I arbitrarily close to S_t with respect to Hausdorff distance between subsets of I. Note that S_t may be viewed as a continuous mapping $[0,1] \to I^{4n} \subset \mathbf{R}^{4n}$. We need to find a continuous map $\widetilde{S}_t\colon [0,1] \to I^{4n}$, which is arbitrarily close to S_t as a subset of I, and such that $\widetilde{S}_t$ does not intersect any of the hyperplanes H_{ij} where the i^{th} and j^{th} coordinates of $\mathbf{R}^{4n}$ are equal, for $1 \leq i < j \leq 4n$. Let $H_{ij}^{\pm}$ be the closed half-spaces determined by H_{ij}. If any segment of S_t lies in H_{ij}^-, we may reflect that with respect to H_{ij} to obtain a mapping $\overline{S}_t$ which is contained in H_{ij}^+ (in other words, whenever two points of S_t cross each other, then we switch their indices). So $\overline{S}_t$ and S_t coincide as subsets of I. Finally, we may perturb $\overline{S}_t$ so that it lies in the interior of H_{ij}^+ which yields the desired mapping $\widetilde{S}_t$. □

Carathéodory's theorem [2] [22, Thm. 1.1.4] states that any point x in the convex hull of a set $X \subset \mathbf{R}^n$ is a convex combination of at most $n+1$ points p_i of X, i.e., there exist constants $a_i > 0$ with $\sum_i a_i = 1$ such that $x = \sum_i a_i p_i$. We need the following continuous version of Carathéodory's theorem:

Lemma 6.2. *Let $p_i\colon [0,1] \to \mathbf{R}^n$, $i = 1, \dots, N$, be $\mathcal{C}^k$ maps. Suppose there exists a $\mathcal{C}^k$ map $x\colon [0,1] \to \mathbf{R}^n$ such that $x(t) \in \operatorname{int}\operatorname{conv}(\{p_i(t)\})$. Then there are $\mathcal{C}^k$ functions $a_i(t)$ such that $a_i > 0$, $\sum_i a_i(t) = 1$, and $\sum_i a_i(t) p_i(t) = x(t)$.*

Proof. We may assume that $x(t) = 0$ after replacing $p_i(t)$ with $p_i(t) - x(t)$. Then we need to find functions $a_i(t)$, such that $P(t)a(t) = 0$, where $P(t)$ is the $n \times N$ matrix with columns $p_i(t)$, and $a(t)$ is the vector with components $a_i(t)$. So $a(t) \in \ker(P(t))$. By Carathéodory's theorem, there are coefficients $a^0(t_0) := (a_1^0(t_0), \dots, a_N^0(t_0))$

fulfilling these properties at any given point $t_0 \in (0,1)$. We may assume that $a_i^0(t_0) > 0$ since 0 lies in the interior of the convex hull. Note that $P(t)$ has constant rank n, since $\operatorname{int}\operatorname{conv}(\{p_i(t)\}) \neq \emptyset$. So $\ker(P(t))$ has dimension $N-n$. Since $p_i(t)$ are $\mathcal{C}^k$, we may choose a $\mathcal{C}^k$ family of orthonormal bases $e_j(t)$ for $\ker(P(t))$ near t_0, by a theorem of Doležal [3, 23, 28]. Then the projection $a^0(t) := \sum \langle a^0(t_0), e_j(t)\rangle$ of $a^0(t_0)$ into $\ker(P(t))$ will be $\mathcal{C}^k$. Furthermore, since $a_i^0(t_0) > 0$, we have $a_i^0(t) > 0$ for t near t_0. So the desired coefficients can be found locally. Thus we may cover $[0,1]$ by a finite number of open subsets where the desired coefficients exist and glue them together via a partition of unity. □

Note 6.3. If $p_i(t)$ in the last result were constant, then the conclusion would follow immediately from a theorem of Kalman [15] who established existence of continuous barycentric coordinates inside convex polytopes.

Note 6.4. Lemma 6.1 does not hold, for any finite number of points, if we require that $x(t)$ lie only in the relative interior of $\operatorname{conv}(f_t)$, i.e., allow f_t to be flat [9].

7. Proof of the Main Results

Here we prove Theorem 4.1, which yields Theorems 2.1 and 1.1 as discussed above. The proof proceeds in four stages: (I) we define a certain point $\overline{x}(t)$ associated to $x(t)$, and select points $p_i(t)$ of f_t which contain $\overline{x}(t)$ within their convex hull; (II) we perturb f_t to a curve $\overline{f}_t$ which contains geodesic segments $\overline{g}_t^i$ near $p_i(t)$, see Figure 2; (III) we perturb $\overline{f}_t$ to the desired homotopy $\widetilde{f}_t$ by replacing $\overline{g}_t^i$ with longer curves $\widetilde{g}_t^i$ via Lemma 5.1; (IV) we verify that $\widetilde{f}_t$ is a $\mathcal{C}^1$-homotopy.

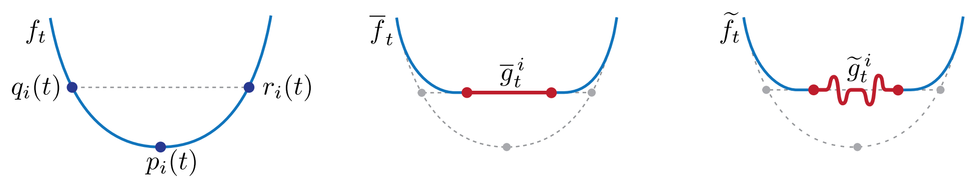

Figure 2.

(I) By the triangle inequality, we may assume that $f_t = f_0$ for $t \in [0,\eta]$, and $f_t = f_1$ for $t \in [1-\eta, 1]$ if $\eta > 0$ is sufficiently small. Accordingly, we set $x(t) = x(0)$ for $t \in [0,\eta]$ and $x(t) = x(1)$ for $t \in [1-\eta, 1]$. So $x(t) = \operatorname{cm}(f_t)$ for $t \in [0,\eta] \cup [1-\eta, 1]$. Define $\overline{x}\colon [0,1] \to \mathbf{R}^n$ by setting $\overline{x}(0) := x(0)$, $\overline{x}(1) := x(1)$, and

$$\overline{x}(t) := \frac{\widetilde{\ell}_t\, x(t) - \ell_t \operatorname{cm}(f_t)}{\widetilde{\ell}_t - \ell_t}$$

for $t \in (0,1)$. Since $x(t) = \operatorname{cm}(f_t)$ near 0 and 1, $\overline{x}$ is continuous. Next note that as $\widetilde{\ell}_t$ grows large, $\overline{x}(t) \to x(t)$. So choosing $\widetilde{\ell}_t$ sufficiently large on $[\eta, 1-\eta]$, we can make sure that $\overline{x}(t) \in \operatorname{int}\operatorname{conv}(f_t)$. Now, by lemma 6.1, we may continuously choose $4n$ distinct points $s_i(t) \in \operatorname{int}(I)$ such that if $p_i(t) := f_t(s_i(t))$ then $\overline{x}(t) \in \operatorname{int}\operatorname{conv}\{p_i(t)\}$.

Thus, by Lemma 6.2, there are continuous functions $a_i(t) > 0$ with $\sum a_i(t) = 1$ such that

$$\sum a_i(t)p_i(t) = \overline{x}(t).$$

We assume that all sums range over $i = 1, \dots, 4n$.

(II) Let $\delta\colon [0,1] \to \mathbf{R}$ be a continuous function with $\delta(t) > 0$ for $t \in (0,1)$ and $\delta(0) = 0 = \delta(1)$. We may assume that $\max(\delta)$ is smaller than the smallest distance in f_t between the points $p_i(t)$, $f_t(a)$, $f_t(b)$. Then there exist points $r_i(t)$, $q_i(t)$ in f_t which are at distance $\delta(t)$ from $p_i(t)$, as measured in f_t, and lie on different sides of $p_i(t)$ when $\delta(t) > 0$. Let $\widehat{r_i(t)q_i(t)}$ be the segment of f_t between $r_i(t)$ and $q_i(t)$. So

$$|\widehat{r_i(t)q_i(t)}| = 2\delta(t).$$

Replacing $\widehat{r_i(t)q_i(t)}$ with the geodesic segment $r_i(t)q_i(t)$ in $\mathbf{S}^{n-1}$, and reparametrizing with constant speed, yields a $\mathcal{C}^0$-homotopy $\widehat{f}_t \in \mathrm{Imm}^0(I, \mathbf{S}^n)$. Since f_t is $\mathcal{C}^1$, $|\widehat{r_i(t)q_i(t)}|/|r_i(t)q_i(t)| \to 1$ as $\delta(t) \to 0$. Thus, if $\max(\delta)$ is sufficiently small, we can make sure that $|r_i(t)q_i(t)| > \delta(t)$. Let $\overline{g}_t^i$ be the subsegment of $r_i(t)q_i(t)$ of length $|\overline{g}_t^i| = a_i(t)\delta(t)$ centered at the midpoint of $r_i(t)q_i(t)$. So $\overline{g}_0^i = p_i(0)$. If we set $\overline{g}_t := \cup_i \overline{g}_t^i$, then we may record that

$$|\overline{g}_t| = \delta(t), \qquad \text{and} \qquad |\overline{g}_t^i| = a_i(t)|\overline{g}_t|.$$

Furthermore, we set

$$\overline{p}_i(t) := \mathrm{cm}(\overline{g}_t^i).$$

Next we smoothen $\widehat{f}_t$ by rounding off its corners at $r_i(t)$ and $q_i(t)$ as follows. Let $I' \subset \mathrm{int}(I)$ be a subsegment so large that $\widehat{f}_t(I')$ contains all segments $r_i(t)q_i(t)$, $\phi\colon I \to \mathbf{R}$ be a $\mathcal{C}^\infty$ (bump) function such that $0 \le \phi \le 1$, $\phi > 0$ on I' and $\phi = 0$ near ∂I, and $\theta_t\colon \mathbf{R} \to \mathbf{R}$ be a continuous family of symmetric $\mathcal{C}^\infty$ functions with $\theta_t \geq 0$, $\int_I \theta_t(s)ds = 1$ and compact support $\mathrm{supp}(\theta_t)$. Now define

$$\overline{f}_t := \frac{\phi(\widehat{f}_t * \theta_t) + (1-\phi)\widehat{f}_t}{|\phi(\widehat{f}_t * \theta_t) + (1-\phi)\widehat{f}_t|}.$$

Then $\overline{f}_t \in \mathrm{Imm}^k(I, \mathbf{S}^{n-1})$, and assuming $\mathrm{supp}(\theta_t)$ is sufficiently small, we have $\overline{g}_t^i \subset \overline{f}_t$. Since $|\widehat{f}_t - f_t|_0 \to 0$ as $\delta(t) \to 0$, 1, $|\widehat{f}_t * \theta_t - f_t|_1 \to 0$, which in turn yields that $|\overline{f}_t - f_t|_1 \to 0$ as $\delta(t) \to 0$, 1. Furthermore, if we set $\overline{\ell}_t := \mathrm{length}(\overline{f}_t)$ and $\widehat{\ell}_t := \mathrm{length}(\widehat{f}_t)$, then $\overline{\ell}_t \to \widehat{\ell}_t$ as $\mathrm{supp}(\theta_t) \to 0$. So assuming $\mathrm{supp}(\theta_t)$ is sufficiently small, we have

$$|\overline{\ell}_t - \ell_t| \le 2|\widehat{\ell}_t - \ell_t| = 2\sum \left(|\widehat{r_i(t)q_i(t)}| - |r_i(t)q_i(t)|\right). \tag{4}$$

(III) Now we construct the desired homotopy $\widetilde{f}_t$ by applying Lemma 5.1 to deform each segment $\overline{g}_t^i$ of $\overline{f}_t$ to a canonical curve $\widetilde{g}_t^i$ with appropriate length $|\widetilde{g}_t^i|$ so that $\mathrm{cm}(\widetilde{f}_t) = x(t)$ and $\mathrm{length}(\widetilde{f}_t) = \widetilde{\ell}_t$. Set $\widetilde{g}_t := \cup_i \widetilde{g}_t^i$. Then $\widetilde{f}_t \setminus \widetilde{g}_t = \overline{f}_t \setminus \overline{g}_t$, which yields

$$|\widetilde{g}_t| = \widetilde{\ell}_t - \overline{\ell}_t + |\overline{g}_t|.$$

Since $\overline{\ell}_t$ can be made arbitrarily close to $\widehat{\ell}_t$ which is smaller than ℓ_t for $t \in (0,1)$, we may assume that $\overline{\ell}_t < \ell_t$ for $t \in (0,1)$ by choosing $\mathrm{supp}(\theta_t)$ sufficiently small, which yields that $|\widetilde{g}_t| > |\overline{g}_t|$ for $t \in (0,1)$. So, by Lemma 5.1, we may require that $\widetilde{g}_t$ and consequently $\widetilde{f}_t$ be nonflat in $\mathbf{S}^{n+m-1}$ for $t \in (0,1)$, as desired. Next note that, by Lemma 5.1, $\mathrm{cm}(\widetilde{g}_t^i) = \mathrm{cm}(\overline{g}_t^i) = \overline{p}_i(t)$. Thus

$$\begin{aligned} \mathrm{cm}(\widetilde{f}_t) &= \sum \frac{|\widetilde{g}_t^i|}{\widetilde{\ell}_t}\,\mathrm{cm}(\widetilde{g}_t^i) + \frac{\widetilde{\ell}_t - |\widetilde{g}_t|}{\widetilde{\ell}_t}\,\mathrm{cm}(\widetilde{f}_t \setminus \widetilde{g}_t) \\ &= \sum \frac{|\widetilde{g}_t^i|}{\widetilde{\ell}_t}\overline{p}_i(t) + \frac{\overline{\ell}_t - |\overline{g}_t|}{\widetilde{\ell}_t}\,\mathrm{cm}(\overline{f}_t \setminus \overline{g}_t). \end{aligned}$$

Setting the last expression above equal to $x(t)$ we obtain

$$\sum \frac{|\widetilde{g}_t^i|}{|\widetilde{g}_t|}\overline{p}_i(t) = \frac{\widetilde{\ell}_t x(t) - \big(\overline{\ell}_t - |\overline{g}_t|\big)\,\mathrm{cm}(\overline{f}_t \setminus \overline{g}_t)}{|\widetilde{g}_t|} \tag{5}$$

for $t \in (0,1)$. Thus $\mathrm{cm}(\widetilde{f}_t) = x(t)$ provided that we find values for $|\widetilde{g}_t^i|$ so that (5) holds. Note that, as $\delta(t) \to 0$, $\mathrm{cm}(\overline{f}_t \setminus \overline{g}_t) \to \mathrm{cm}(f_t)$, $\overline{\ell}_t - |\overline{g}_t| \to \ell_t$, and $|\widetilde{g}_t| \to \widetilde{\ell}_t - \ell_t$. So the right hand side of (5) which we call $\widetilde{x}(t)$ converges to $\overline{x}(t)$, and thus lies in $\mathrm{int\,conv}\{p_i(t)\}$ if $\max(\delta)$ is sufficiently small. On the other hand $\overline{p}_i(t) \to p_i(t)$. So assuming $\max(\delta)$ is sufficiently small, we have

$$\widetilde{x}(t) \in \mathrm{int\,conv}\{\overline{p}_i(t)\}$$

for $t \in (0,1)$. If we set $\widetilde{x}(0) := x(0)$, $\widetilde{x}(1) := x(1)$, then $\widetilde{x}$ is continuous on $[0,1]$, and the above inclusion still holds at $t = 0,\ 1$, since $\overline{p}_i(0) = \overline{g}_0^i = p_i(0)$ and $\overline{p}_i(1) = \overline{g}_1^i = p_i(1)$ by definition. Thus, by Lemma 6.2, there are continuous functions $\widetilde{a}_i(t) > 0$ with $\sum \widetilde{a}_i(t) = 1$ such that $\sum \widetilde{a}_i(t)\overline{p}_i(t) = \widetilde{x}(t)$. Hence if we set

$$|\widetilde{g}_t^i| := \widetilde{a}_i(t)|\widetilde{g}_t|,$$

then (5) holds and we obtain $\mathrm{cm}(\widetilde{f}_t) = x(t)$ as desired.

(IV) It remains to check that $\widetilde{f}_t$ is a $\mathcal{C}^1$-homotopy, i.e., $|\widetilde{f}_t - f_0|_1 \to 0$ as $t \to 0$. To this end note that $|\widetilde{f}_t - f_0|_1 \leq |\widetilde{f}_t - \overline{f}_t|_1 + |\overline{f}_t - f_0|_1$ and $|\overline{f}_t - f_0|_1 \to 0$ as we discussed in Part II above. So we just need to show that $|\widetilde{f}_t - \overline{f}_t|_1 \to 0$, which is the case provided that $|\widetilde{g}_t^i - g_t^i|_1 \to 0$. Thus, by Lemma 5.1, it suffices to check that $|\widetilde{g}_t^i|/|\overline{g}_t^i| \to 1$, as $t \to 0,\ 1$. Note that, for $t \in (0,1)$,

$$\frac{|\widetilde{g}_t^i|}{|\overline{g}_t^i|} = \frac{\widetilde{a}_i(t)|\widetilde{g}_t|}{a_i(t)|\overline{g}_t|} = \frac{\widetilde{a}_i(t)}{a_i(t)}\left(\frac{|\widetilde{\ell}_t - \overline{\ell}_t|}{\delta(t)} + 1\right).$$

Furthermore, we have

$$\frac{\widetilde{\ell}_t - \overline{\ell}_t}{\delta(t)} \leq \frac{|\widetilde{\ell}_t - \ell_t|}{\delta(t)} + \frac{|\ell_t - \overline{\ell}_t|}{\delta(t)}.$$

We can make sure that $(\widetilde{\ell}_t - \ell_t)/\delta(t)$ vanishes as $t \to 0,\ 1$, by setting $\delta(t) \geq (\widetilde{\ell}_t - \ell_t)^{1/2}$ for t close to 0, 1. This is possible since $\widetilde{\ell}_0 = \ell_0$, $\widetilde{\ell}_1 = \ell_1$, and all other constraints for $\delta(t)$ mentioned above where positive upper bounds for $\max(\delta)$. Next note that by (4),

$$\frac{\ell_t - \overline{\ell}_t}{\delta(t)} = 2\frac{\ell_t - \overline{\ell}_t}{|\widehat{r_i(t)q_i(t)}|} \leq 4\sum\left(1 - \frac{|r_i(t)q_i(t)|}{|\widehat{r_i(t)q_i(t)}|}\right).$$

Since $\delta(0) = 0 = \delta(1)$, $|\widehat{r_i(t)q_i(t)}| \to 0$ as $t \to 0, 1$. Thus, since f_t is $\mathcal{C}^1$, the right hand side of the last expression above vanishes as $t \to 0, 1$. Hence $|\widetilde{g}_t|/|\overline{g}_t| \to 1$. Now it suffices to note that $\widetilde{a}_i(t_0) = a_i(t_0)$, for $t_0 = 0, 1$. Indeed by definition $\widetilde{a}_i(t_0)$ are barycentric coordinates of $\widetilde{x}(t_0) = x(t_0) = \overline{x}(t_0)$, with respect to $p_i(t_0)$. Furthermore, $a_i(t)$ are the coordinates of $\overline{x}(t)$ with respect to $p_i(t)$, which completes the proof.

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