AFFINE UNFOLDINGS OF CONVEX POLYHEDRA: PROGRESS ON DÜRER'S PROBLEM

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Convex polyhedra are among the oldest mathematical objects. Indeed the five platonic solids, which constitute the climax of Euclid's books, were already known to the ancient people of Scotland some 4000 years ago [1]. During the Renaissance, polyhedra were once again objects of fascination while painters were discovering the rules of perspective and laying the foundations of projective geometry. This remarkable confluence of art and mathematics was personified in a number of highly creative individuals including the German painter Albrecht Dürer who was based in Nüremberg at the dawn of the 16^{th} century, and is credited with ushering the advent of Renaissance in Northern Europe. During extended trips over the Alps, Dürer learned the rules of perspective from his Italian contemporaries, and subsequently described them in his influential book, *The Painter's Manual* [4]. Aside from being the first geometry text published in German, this work is remarkable for containing the first recorded examples of unfoldings of polyhedra.

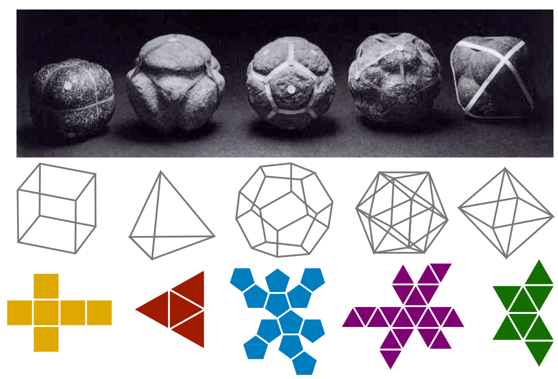

FIGURE 1. First Row: Neolithic carved stones from 2000BC discovered in Scotland. Second Row: The familiar representations of platonic solids studied in Euclid's *Elements*. Third Row: Examples of unfoldings of the platonic solids.

An (edge) unfolding of a polyhedron is the process of cutting it along a collection of its edges, without disconnecting it, so that the resulting surface may be developed isometrically into the plane. Many school children are familiar with the process of cutting out a template from craft books, and folding the paper along dotted lines to form simple polyhedra such as a tetrahedron or a cube; an unfolding is the reverse process. Note that the cuts are made along a connected subset of the edges of P which contains each vertex of P and no closed paths. In other words, the cut set forms a *spanning tree* of the edge graph of P, and thus a convex polyhedron admits many different unfoldings depending on the choice of this tree. Furthermore, it is not the case that every unfolding of every polyhedron is simple or non overlapping. For instance there are even some (non regular) tetrahedra, which admit some unfoldings which overlap themselves. On the other hand, all the examples of unfoldings

FIGURE 2. A self-portrait of Dürer completed in the year 1500 at the age of 28, together with some illustrations from his book, The Painter's Manual.

which Dürer constructed were simple, and in the intervening 5 centuries no one has yet discovered a convex polyhedron which does not admit some simple unfolding.

The problem of existence of simple unfoldings for convex polyhedra were explicitly posed in 1970's by Shephard [6], and the assertion that a solution can always be found, or that every convex polyhedron is unfoldable (in one-to-one fashion) has been dubbed Dürer's conjecture. There is, however, substantial empirical evidence both for and against this supposition. On the one hand, computers have found simple unfoldings for countless convex polyhedra through an exhaustive search of their spanning edge trees. On the other hand, there is still no algorithm for finding the right tree, and computer experiments suggest that the probability that a random edge unfolding of a generic polyhedron overlaps itself approaches 1 as the number of vertices grow [3]. To date the problem remains wide open, and it is not even known whether simple classes of polyhedra such as prismatoids (polyhedra generated by the convex hull of a pair of convex polygons in parallel planes) are unfoldable.

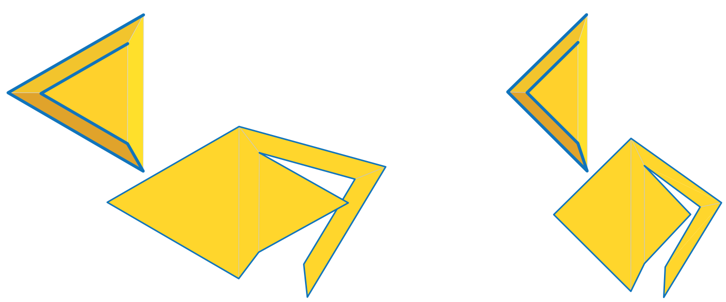

FIGURE 3. The left side shows a truncated tetrahedron (viewed from "above") together with an overlapping unfolding of it generated by a monotone edge tree. As we see on the right side, however, the same edge tree generates a simple unfolding once the polyhedron has been stretched.

Recently the author has been able to make some progress in this area by solving a weaker form of Dürer's problem posed by Croft, Falconer, and Guy [2, B21]: is every convex

Date: Last Typeset January 31, 2015.

Research of the author was supported in part by NSF Grants DMS-0336455, DMS-1308777, and Simons Collaboration Grant 279374.

polyhedron combinatorially equivalent to an unfoldable one? It turns out that the answer is yes, and therefore there exists no combinatorial obstruction to a positive resolution of Dürer's problem. What the author shows is that every convex polyhedron becomes unfoldable after an affine (or linear) transformation. More explicitly, suppose that a convex polyhedron P is in general position in $\mathbf{R}^3$, i.e., no two of its vertices are at the same height. Then it is easy to construct a spanning tree T of P which is *monotone*, i.e., if T is rooted at the lowest vertex r of P, then each of the branches of T which connect its leaves to r have strictly decreasing heights or z-coordinates. Now stretch P via a rescaling along the z-axis. Then the corresponding unfolding eventually becomes simple, as illustrated in Figure 3. The proof that this stretching procedure works is by induction on the number of leaves (or branches of T which connect each leaf to the root r). The first step, i.e., when T consists of only one branch, is relatively simple to prove and follows from a topological characterization for embeddings among immersed disks in the plane. The inductive step is more technical. See the author's paper [5] for further details and references.

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