CONTINUITY OF TOTAL CURVATURES OF RIEMANNIAN HYPERSURFACES

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Abstract. We show that total generalized mean curvatures of hypersurfaces with positive reach in Riemannian manifolds, and convex bodies in Cartan-Hadamard spaces, are continuous with respect to Hausdorff distance.

1. Introduction

Let Γ be a $\mathcal{C}^{1,1}$ hypersurface in a Riemannian n-manifold M, cooriented by a (continuous) unit normal vector field ν. Then the principal curvatures $\kappa := (\kappa_1, \dots, \kappa_{n-1})$ of Γ with respect to ν are well-defined almost everywhere, by Rademacher's theorem, and the *total* r^{th} *mean curvature* of Γ is given by

$$\mathcal{M}_r(\Gamma) := \int_\Gamma \sigma_r(\kappa),$$

where $\sigma_r(\kappa) := \sum_{1\leq i_1<\dots<i_r\leq n-1} \kappa_{i_1}\dots\kappa_{i_r}$ are the elementary symmetric polynomials, for $1 \leq r \leq n-1$. By convention, $\sigma_0 := 1$, and $\sigma_r := 0$ for $r \geq n$. Up to multiplicative constants, depending only on n, $\mathcal{M}_r(\Gamma)$ form the coefficients of the generalized Steiner's polynomial and Weyl's tube formula [14]. They are also known as quermassintegrals when Γ is a convex hypersurface in Euclidean space $\mathbf{R}^n$ [16]. Here we study the continuity of these fundamental objects. In particular, we show:

Theorem 1.1. *Let Γ be a closed cooriented hypersurface with positive reach embedded in a Riemannian manifold M. Suppose there exists a sequence of closed embedded hypersurfaces $\Gamma_i \subset M$ with uniformly positive reach, and coorientations consistent with that of Γ, such that $\Gamma_i \to \Gamma$ with respect to Hausdorff distance. Then $\mathcal{M}_r(\Gamma_i) \to \mathcal{M}_r(\Gamma)$.*

The *reach* of Γ, denoted by $\mathrm{reach}(\Gamma)$, is the supremum of $\varepsilon \geq 0$ such that through each point of Γ there pass a pair of (geodesic) balls of radius ε whose interiors are disjoint from Γ. If $\mathrm{reach}(\Gamma) > 0$, then Γ is $\mathcal{C}^{1,1}$ [11, Lem. 2.6]. Thus $\mathcal{M}_r(\Gamma)$ is well-defined. If $\Gamma_i \to \Gamma$ with respect to Hausdorff distance, then $\Gamma_i \to \Gamma$ in $\mathcal{C}^1$-topology (Lemma 3.1). In particular, if Γ is cooriented by the unit normal vector field ν, then we may choose unit normal vector fields ν_i along Γ_i such that $\nu_i \to \nu$ in local coordinates. This is what we mean by the assumption that the coorientations of Γ_i are *consistent* with that of Γ.

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Theorem 1.1 was established in $\mathbf{R}^n$ by Federer [7, Thm. 5.9]. The general version should follow from the theory of smooth valuations [1], and convergence of normal cycles [8, 17]. The latter can be reduced to the Euclidean case [7, Thm. 4.13] via local charts, since positive reach is preserved under diffeomorphisms [2]. Here we give a more direct and fairly self-contained argument via universal differential forms introduced by Chern [6].

The prime motivation for this work is the next observation, which we again establish geometrically using Theorem 1.1 and some recent results for total curvatures [13]. A *Cartan-Hadamard manifold* M is a complete, simply connected manifold with nonpositive curvature. A subset of M is *convex* if it contains the geodesic connecting every pair of its points. A *convex hypersurface* $\Gamma \subset M$ is the boundary of a compact convex set with interior points, which we assume to be cooriented by the outward normal. We define

$$\mathcal{M}_r(\Gamma) := \lim_{\varepsilon \searrow 0} \mathcal{M}_r(\Gamma^\varepsilon), \tag{1}$$

where Γ^ε denotes the outer parallel hypersurface of Γ at distance ε. Note that $\mathcal{M}_r(\Gamma^\varepsilon)$ is well-defined since $\mathrm{reach}(\Gamma^\varepsilon) \geq \varepsilon$ and thus Γ^ε is $\mathcal{C}^{1,1}$ for $\varepsilon > 0$ [11, Lem. 2.6]. Furthermore, the limit exists since

$$\varepsilon \mapsto \mathcal{M}_r(\Gamma^\varepsilon) \ \text{ is nondecreasing,} \tag{2}$$

by [13, Cor. 4.4], and $\mathcal{M}_r(\Gamma^\varepsilon) \geq 0$ since Γ^ε is convex (and cooriented by the outward normal). See [11] for basic facts about convex sets and their distance functions in Cartan-Hadamard manifolds.

Theorem 1.2. *The total curvature functionals $\mathcal{M}_r$ are continuous on the space of convex hypersurfaces in a Cartan-Hadamard manifold with respect to Hausdorff distance.*

This result simplifies a number of arguments, e.g., see [11, Note 3.7] and [12, Lem. 3.3], related to the *Cartan-Hadamard conjecture* on the isoperimetric inequality in spaces of nonpositive curvature. The conjecture follows if the *total Gauss-Kronecker curvature*

$$\mathcal{M}_{n-1}(\Gamma) \geq |\mathbf{S}^{n-1}| \tag{3}$$

for convex hypersurfaces Γ in Cartan-Hadamard manifolds, where $|\mathbf{S}^{n-1}|$ is the volume of the unit sphere in $\mathbf{R}^n$ [11]. Our proof of Theorem 1.2 employs an estimate for total curvatures of parallel hypersurfaces (Lemma 4.1) which may be of further interest.

2. UNIVERSAL DIFFERENTIAL FORMS

Let M be a Riemannian manifold, and $\Gamma \subset M$ be a hypersurface cooriented by a unit normal vector field ν. Let T_pM be the tangent space of M at a point p, $S_p \subset T_pM$ be the set of unit vectors, and $SM := \{(p,u) \mid p \in M, u \in S_p\}$ denote the unit tangent bundle of M. Let $\overline{\nu}\colon \Gamma \to SM$ be given by $\overline{\nu}(p) := (p, \nu(p))$. The following fact is established in [3, Prop. 3.8].

Lemma 2.1 (Bernig-Bröcker [3])**.** *For* $0 \leq r \leq n-1$*, there exists an* $(n-1)$*-form* Φ_r *on* SM *such that*

$$\mathcal{M}_r(\Gamma) = \int_\Gamma \overline{\nu}^*(\Phi_r),$$

for any $\mathcal{C}^{1,1}$ *hypersurface* $\Gamma \subset M$*.*

The forms Φ_r are called *universal* [3] because they do not depend on Γ. The form Φ_{n-1} corresponds to Φ_0 in Chern [6, p. 675], and is also described by Borbely [4]. See [10] for a concise construction of Φ_r in terms of the connection forms of M and dual one forms of the principal frame of Γ.

We describe a geometric construction for $\Phi := \Phi_{n-1}$, which avoids exterior algebra. This is of special interest in connection with conjecture (3). Let $GK := \sigma_{n-1}(\kappa)$ be the *Gauss-Kronecker curvature* of Γ. To motivate our approach, note that in $\mathbf{R}^n$

$$\nu^*(\mathrm{dvol}_{\mathbf{S}^{n-1}}) = GK\mathrm{dvol}_\Gamma,$$

where dvol stands for volume form. So the Gauss-Kronecker curvature of Γ is the Jacobian of ν viewed as the Gauss map $\Gamma \to \mathbf{S}^{n-1}$. Hence we may set $\Phi := \mathrm{dvol}_{\mathbf{S}^{n-1}}$. To extend this concept to M, note that each tangent space of SM admits a decomposition [15, Sec. 1.3] into "horizontal" and "vertical" components given by

$$T_{(p,v)}SM = T_pM \oplus (v^\perp) \simeq T_pM \oplus T_vS_p,$$

where $v^\perp \subset T_pM$ is the subspace orthogonal to v, which may be identified with T_vS_p by parallel transport within T_pM. Let $\pi\colon T_{(p,v)}SM \to T_vS_p$ be projection onto the vertical component. Then we set

$$\Phi_{(p,v)} := \pi^*(\mathrm{dvol}_{S_p})_v.$$

To check this construction, let $e_i \in T_p\Gamma$ be an orthonormal set of principal directions with corresponding curvatures κ_i. Note that $d\pi = \pi$ and $d\overline{\nu}(e_i) = (e_i, \nabla_{e_i}\nu)$, where ∇ is the covariant derivative. Thus $d(\pi \circ \overline{\nu})(e_i) = \pi(e_i, \nabla_{e_i}\nu) = \nabla_{e_i}\nu = \kappa_i e_i$. So we have

$$\begin{aligned}(\overline{\nu}^*\Phi)_p(e_1, \dots, e_{n-1}) &= (\pi \circ \overline{\nu})^*(\mathrm{dvol}_{S_p})_{\nu(p)}(e_1, \dots, e_{n-1}) \\ &= (\mathrm{dvol}_{S_p})_{\nu(p)}(\kappa_1 e_1, \dots, \kappa_{n-1}e_{n-1}) = GK\,(\mathrm{dvol}_\Gamma)_p(e_1, \dots, e_{n-1}).\end{aligned}$$

Hence $\overline{\nu}^*\Phi = GK\,\mathrm{dvol}_\Gamma$, as desired.

3. PROOF OF THEOREM 1.1

Let ν be the unit normal vector field coorienting Γ, and $u\colon M \to \mathbf{R}$ be the signed distance function of Γ with respect to ν. For $0 < \delta < \mathrm{reach}(\Gamma)$, let $U := u^{-1}([-\delta, \delta])$ be the *tubular neighborhood* of Γ of radius δ. Then $u \in \mathcal{C}^{1,1}(U)$ [11, Lem. 2.6], which means that u is $\mathcal{C}^{1,1}$ in a collection of local coordinate charts of M covering U. Fix $\delta < \min\{\mathrm{reach}(\Gamma)/2, \mathrm{reach}(\Gamma_i)/2\}$. We assume i is so large that $\Gamma_i \subset U$.

First assume that Γ_i do not intersect Γ. Let Ω_i be the compact region between Γ and Γ_i in U. Choose a unit normal vector field ν_i along Γ_i so that ν_i points into (away from) Ω_i, if ν points away from (into) Ω_i. Let u_i be the signed distance function of Γ_i with respect to ν_i. Note that $u_i \in \mathcal{C}^{1,1}(U)$, since $\delta < \mathrm{reach}(\Gamma_i)/2$.

Lemma 3.1. *$u_i \to u$ in $\mathcal{C}^1(U)$, and ∇u_i are uniformly Lipschitz on U.*

Proof. By [11, Prop. 2.8], the Hessians of u and u_i are uniformly bounded almost everywhere on U. Hence, on U, the gradients ∇u and ∇u_i are uniformly Lipschitz. Next we show that $u_i \to u$ in $\mathcal{C}^1(U)$. For any point $p \in U \setminus \Gamma$ there exists a (geodesic) sphere $S \subset U$ of radius $|u(p)| > 0$ centered at p with $S \cap \Gamma = \{\overline{p}\}$. Similarly, assuming i is so large that $p \notin \Gamma_i$, there exists a sphere $S_i \subset U$ of radius $|u_i(p)| > 0$ centered at p with $S_i \cap \Gamma_i = \{\overline{p}_i\}$. Since $\Gamma_i \to \Gamma$ in Hausdorff distance, $u_i \to u$ in $\mathcal{C}^0(U)$. Thus

$$\mathrm{dist}(\overline{p}_i, S) \leq \mathrm{dist}(S_i, S) \to 0, \qquad \text{and} \qquad \mathrm{dist}(\overline{p}_i, \Gamma) \leq \mathrm{dist}(\Gamma_i, \Gamma) \to 0,$$

where dist is the Riemannian distance in M. Thus any limit point of $\overline{p}_i$ lies in $\Gamma \cap S = \{\overline{p}\}$, or $\overline{p}_i \to \overline{p}$. It follows that $\nabla u_i(p) \to \nabla u(p)$, since these gradients are unit tangent vectors at p to geodesic segments $p\overline{p}$ and $p\overline{p}_i$. Since the gradients are uniformly Lipschitz, and $U \setminus \Gamma$ is dense in U, $\nabla u_i(p) \to \nabla u(p)$ for all $p \in U$. Finally, since U has compact closure, and ∇u_i are uniformly Lipschitz, it follows that $\nabla u_i \to \nabla u$ uniformly on U, which completes the proof. $\square$

As pointed out in the introduction, the above lemma implies that $\nu_i \to \nu$ in local coordinates (so the coorientations of Γ_i are consistent with that of Γ). The above lemma together with Kirszbraun's extension theorem leads to:

Lemma 3.2. *There exists $N > 0$ such that, for $i > N$, ν_i extends to a uniformly Lipschitz unit vector field on Ω_i which coincides with ν on Γ.*

Proof. Suppose first that U is parallelizable, so that there exists a smooth orthonormal frame field e_j on U. Then any unit vector in T_pU is identified with a point of $\mathbf{S}^{n-1}$ by

$$v \mapsto \big(\langle v, e_1\rangle, \dots, \langle v, e_n\rangle\big),$$

where $\langle\cdot,\cdot\rangle$ denotes the metric on M. In particular, the union of ν and ν_i yields a mapping $w_i\colon \partial\Omega_i \to \mathbf{S}^{n-1}$. By Lemma 3.1, w_i are uniformly Lipschitz, say with constant L. So by Kirszbraun's theorem, w_i admits an L-Lipschitz extension $\Omega_i \to \mathbf{R}^n$, which we again denote by w_i. For any point $p \in \Omega_i$ let $\overline{p} \in \Gamma$ be its nearest point. Then $\mathrm{dist}(p, \overline{p}) \leq \delta_i$, where δ_i denotes the maximum length in Ω_i of the geodesics orthogonal to Γ. Assume N is so large that $\delta_i < 1/L$. Then, by the triangle inequality,

$$|w_i(p)| \geq |w_i(\overline{p})| - |w_i(p) - w_i(\overline{p})| \geq 1 - L\delta_i > 0.$$

Hence $\nu_i := w_i/|w_i|$ yields the desired extension.

If U is not parallelizable, cover Γ by open topological balls $B_k \subset \Gamma$. Let $U_k \subset U$ be the cylindrical neighborhoods foliated by geodesics in U which are orthogonal to B_k. Then U_k admits a smooth orthonormal frame field. So, as discussed above, there exists a unit normal vector field ν_i^k on $\Omega_i \cap U_k$ which is L-Lipschitz, and coincides with ν and ν_i on $\partial\Omega_i \cap U_k$. Let ϕ_k be a partition of unity on U subordinate to $\{U_k\}$, and set

$$\nu_i := \sum_k \phi_k \nu_i^k \Big/ \Big| \sum_k \phi_k \nu_i^k \Big|.$$

We claim that for N sufficiently large, $\sum_k \phi_k \nu_i^k \neq 0$ and thus ν_i is well-defined. Indeed for any point $p \in \Omega_i \cap U_k$, we have $\overline{p} \in \Gamma \cap U_k$. Thus

$$|\nu_i^k(p) - \nu(\overline{p})| = |\nu_i^k(p) - \nu_i^k(\overline{p})| \leq L\delta_i.$$

Consequently, if $p \in \Omega_i \cap U_k \cap U_\ell$, then

$$|\nu_i^k(p) - \nu_i^\ell(p)| \leq |\nu_i^k(p) - \nu(\overline{p})| + |\nu(\overline{p}) - \nu_i^\ell(\overline{p})| \leq 2L\delta_i.$$

So if N is sufficiently large, $|\nu_i^k(p) - \nu_i^\ell(p)| \leq 1$; in particular, these vectors all lie in an open hemisphere. Hence $\sum \phi_k \nu_i^k \neq 0$, which ensures that ν_i is the desired extension. □

Let ν_i be the extension given by Lemma 3.2. Then by Lemma 2.1 and Stokes theorem, there exists a constant C independent of i such that

$$|\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma)| = \left| \int_{\partial\Omega_i} \overline{\nu}_i^*(\Phi_r) \right| = \left| \int_{\Omega_i} d(\overline{\nu}_i^*(\Phi_r)) \right| = \left| \int_{\Omega_i} \overline{\nu}_i^*(d\Phi_r) \right| \leq C|\Omega_i|, \tag{4}$$

since ν_i are uniformly Lipschitz, and so the pullbacks $\overline{\nu}_i^*$ are uniformly bounded. Thus $|\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma)| \to 0$, which concludes the proof in the case where $\Gamma_i \cap \Gamma = \emptyset$.

To prove the general case, let $\Gamma' \subset U$ be a hypersurface parallel to Γ, i.e., a level set of u different from Γ. Let Ω' be the region between Γ' and Γ. Then Γ_i will be disjoint from Γ' for i sufficiently large. Let Ω_i' be the region between Γ' and Γ_i. Since Γ, Γ', and Γ_i all have uniformly positive reach, the same argument for (4) shows that

$$|\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma')| \leq C|\Omega_i'|, \qquad \text{and} \qquad |\mathcal{M}_r(\Gamma') - \mathcal{M}_r(\Gamma)| \leq C|\Omega'|,$$

for some constant C. Thus, by the triangle inequality,

$$\lim_{i\to\infty} |\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma)| \leq \lim_{i\to\infty} C(|\Omega_i'| + |\Omega'|) \leq 2C|\Omega'|.$$

As $\Gamma' \to \Gamma$, we have $|\Omega'| \to 0$, which completes the proof.

Note 3.3. The above proof shows that Theorem 1.1 holds for any functional on the space of closed embedded hypersurfaces $\Gamma \subset M$ with bounded reach which is obtained by integrating a universal differential form as in Lemma 2.1. In particular, these functionals include boundary terms of Lipschitz-Killing curvatures; see Chern [6], Cheeger–Müller–Schrader [5], and Fu–Wannerer [9].

4. PROOF OF THEOREM 1.2

Let M be a Cartan-Hadamard manifold and Γ be a convex hypersurface in M. Recall that Γ^ε denote the outer parallel hypersurfaces of Γ at distance $\varepsilon \geq 0$. Let Ω_ε be the region between Γ^ε and Γ, and K_M denote the sectional curvature of M.

Lemma 4.1. *Let* $C := \sup_{\Omega_\varepsilon} |K_M|$. *Then*

$$|\mathcal{M}_r(\Gamma^\varepsilon) - \mathcal{M}_r(\Gamma)| \leq \big((r+1)\mathcal{M}_{r+1}(\Gamma^\varepsilon) + C\mathcal{M}_{r-1}(\Gamma^\varepsilon)\big)\varepsilon.$$

Proof. Let u be the distance function of Γ. Then $|\nabla u| = 1$ on the exterior region of Γ, and [13, Thm. 3.1] quickly yields

$$\mathcal{M}_r(\Gamma^\varepsilon) - \mathcal{M}_r(\Gamma) \leq (r+1)\int_{\Omega_\varepsilon} \sigma_{r+1}(\kappa^u) + C\int_{\Omega_\varepsilon} \sigma_{r-1}(\kappa^u),$$

where $\kappa^u = (\kappa^u_1, \dots, \kappa^u_{n-1})$ refers to the principal curvatures of the level sets of u. More precisely, we apply [13, Thm. 3.1] to parallel hypersurfaces Γ^δ for $0 < \delta < \varepsilon$ and take the limit as $\delta \to 0$ to obtain the above inequality. By the coarea formula

$$\int_{\Omega_\varepsilon} \sigma_{r+1}(\kappa^u) = \int_{0 \leq t \leq \varepsilon} \mathcal{M}_{r+1}(\Gamma^t) \leq \varepsilon \mathcal{M}_{r+1}(\Gamma^\varepsilon),$$

where the last inequality is due to the monotonicity property (2). Similarly, $\int_{\Omega_\varepsilon} \sigma_{r-1}(\kappa^u) \leq \varepsilon\mathcal{M}_{r-1}(\Gamma^\varepsilon)$, which completes the proof. $\square$

Suppose there exists a sequence of convex hypersurfaces $\Gamma_i \subset M$ such that $\Gamma_i \to \Gamma$ with respect to Hausdorff distance. By the triangle inequality,

$$|\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma)| \leq \\ |\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma_i^\varepsilon)| + |\mathcal{M}_r(\Gamma_i^\varepsilon) - \mathcal{M}_r(\Gamma^\varepsilon)| + |\mathcal{M}_r(\Gamma^\varepsilon) - \mathcal{M}_r(\Gamma)|.$$

As $i \to \infty$, the middle term on the right hand side vanishes by Theorem 1.1. To bound the first term, let $B \subset M$ be a closed ball which contains Γ in its interior, and set $C := \sup_B |K_M|$. For i sufficiently large and small ε, we have Γ_i, $\Gamma_i^\varepsilon \subset B$; therefore, Lemma 4.1 yields

$$|\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma_i^\varepsilon)| \leq \big((r+1)\mathcal{M}_{r+1}(\Gamma_i^\varepsilon) + C\mathcal{M}_{r-1}(\Gamma_i^\varepsilon)\big)\varepsilon.$$

But $\mathcal{M}_{r+1}(\Gamma_i^\varepsilon) \to \mathcal{M}_{r+1}(\Gamma^\varepsilon)$ and $\mathcal{M}_{r-1}(\Gamma_i^\varepsilon) \to \mathcal{M}_{r-1}(\Gamma^\varepsilon)$ by Theorem 1.1, since these hypersurfaces have uniformly positive reach. Thus

$$\lim_{i\to\infty} |\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma)| \leq \big((r+1)\mathcal{M}_{r+1}(\Gamma^\varepsilon) + C\mathcal{M}_{r-1}(\Gamma^\varepsilon)\big)\varepsilon + |\mathcal{M}_r(\Gamma^\varepsilon) - \mathcal{M}_r(\Gamma)|.$$

Letting $\varepsilon \to 0$ and recalling (1) completes the proof.

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