CONTINUITY OF TOTAL CURVATURES OF RIEMANNIAN HYPERSURFACES

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ABSTRACT. We show that total generalized mean curvatures of hypersurfaces with positive reach in Riemannian manifolds, and convex bodies in Cartan-Hadamard spaces, are continuous with respect to Hausdorff distance.

1. Introduction

Let Γ be an oriented $\mathcal{C}^{1,1}$ hypersurface in a Riemannian *n*-manifold M. Then the principal curvatures $\kappa := (\kappa_1, \ldots, \kappa_{n-1})$ of Γ are well-defined almost everywhere, by Rademacher's theorem, and the *total* r^{th} mean curvature of Γ is given by

$$\mathcal{M}_r(\Gamma) := \int_{\Gamma} \sigma_r(\kappa),$$

where $\sigma_r(\kappa) := \sum_{1 \leq i_1 < \dots < i_r \leq n-1} \kappa_{i_1} \dots \kappa_{i_r}$ are the elementary symmetric polynomials, for $1 \leq r \leq n-1$. By convention, $\sigma_0 := 1$, and $\sigma_r := 0$ for $r \geq n$. Up to multiplicative constants, depending only on n, $\mathcal{M}_r(\Gamma)$ form the coefficients of the generalized Steiner's polynomial and Weyl's tube formula [12]. They are also known as quermassintegrals when Γ is a convex hypersurface in Euclidean space \mathbf{R}^n [14]. Here we study the continuity of these fundamental objects. In particular, we show:

Theorem 1.1. Let Γ be a closed hypersurface with positive reach embedded in a Riemannian manifold M. Suppose there exists a sequence of closed embedded hypersurfaces $\Gamma_i \subset M$ with uniformly positive reach such that $\Gamma_i \to \Gamma$ with respect to Hausdorff distance. Then $\mathcal{M}_r(\Gamma_i) \to \mathcal{M}_r(\Gamma)$.

The reach of Γ , denoted by reach(Γ), is the supremum of $\varepsilon \geq 0$ such that through each point of Γ there pass a pair of (geodesic) balls of radius ε whose interiors are disjoint from Γ . If reach(Γ) > 0, then Γ is $\mathcal{C}^{1,1}$ [9, Lem. 2.6]. Thus $\mathcal{M}_r(\Gamma)$ is well-defined. Theorem 1.1 was established in \mathbf{R}^n by Federer [6, Thm. 5.9]. The general version should follow from the theory of smooth valuations [1], and convergence of normal cycles [7,15]. The latter can be reduced to the Euclidean case [6, Thm. 4.13] via local charts, since positive reach is preserved under diffeomorphisms [2]. Here we give a more direct and fairly self-contained argument via universal differential forms introduced by Chern [5].

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The prime motivation for this work is the next observation, which we again establish geometrically using Theorem 1.1 and some recent results for total curvatures [11]. A Cartan-Hadamard manifold M is a complete, simply connected manifold with nonpositive curvature. A subset of M is convex if it contains the geodesic connecting every pair of its points. A convex hypersurface $\Gamma \subset M$ is the boundary of a compact convex set with interior points. We define

(1)
$$\mathcal{M}_r(\Gamma) := \lim_{\varepsilon \searrow 0} \mathcal{M}_r(\Gamma^{\varepsilon}),$$

where Γ^{ε} denotes the outer parallel hypersurface of Γ at distance ε . Note that $\mathcal{M}_r(\Gamma^{\varepsilon})$ is well-defined since reach $(\Gamma^{\varepsilon}) \geq \varepsilon$ and thus Γ^{ε} is $\mathcal{C}^{1,1}$ for $\varepsilon > 0$. Furthermore, the limit exists since

(2)
$$\varepsilon \mapsto \mathcal{M}_r(\Gamma^{\varepsilon})$$
 is nondecreasing,

by [11, Cor. 4.4], and $\mathcal{M}_r(\Gamma^{\varepsilon}) \geq 0$ since Γ^{ε} is convex (assuming proper orientation, so that the principal curvatures are nonnegative). See [9] for basic facts about convex sets and their distance functions in Cartan-Hadamard manifolds.

Theorem 1.2. The total curvature functionals \mathcal{M}_r are continuous on the space of convex hypersurfaces in a Cartan-Hadamard manifold with respect to Hausdorff distance.

This result simplifies a number of arguments, e.g., see [9, Note 3.7] and [10, Lem. 3.3], related to the *Cartan-Hadamard conjecture* on the isoperimetric inequality in spaces of nonpositive curvature. The conjecture follows if the *total Gauss-Kronecker curvature*

(3)
$$\mathcal{M}_{n-1}(\Gamma) \ge |\mathbf{S}^{n-1}|$$

for convex hypersurfaces Γ in Cartan-Hadamard manifolds, where $|\mathbf{S}^{n-1}|$ is the volume of the unit sphere in \mathbf{R}^n [9]. Our proof of Theorem 1.2 employs an estimate for total curvatures of parallel hypersurfaces (Lemma 4.1) which may be of further interest.

2. Universal Differential Forms

Let T_pM be the tangent space of a Riemannian manifold M at a point $p, S_p \subset T_pM$ be the set of unit vectors, and $SM := \{(p,u) \mid p \in M, u \in S_p\}$ denote the unit tangent bundle of M. Let ν be a (continuous) unit normal vector field along Γ , and $\overline{\nu} \colon \Gamma \to SM$ be given by $\overline{\nu}(p) := (p, \nu(p))$. The following fact is established in [3, Prop. 3.8].

Lemma 2.1 (Bernig-Bröcker [3]). For $0 \le r \le n-1$, there exists an (n-1)-form Φ_r on SM such that

$$\mathcal{M}_r(\Gamma) = \int_{\Gamma} \overline{\nu}^*(\Phi_r),$$

for any $C^{1,1}$ hypersurface $\Gamma \subset M$.

The forms Φ_r are called *universal* [3] because they do not depend on Γ . The form Φ_{n-1} corresponds to Φ_0 in Chern [5, p. 675], and is also described by Borbely [4]. See [8] for a concise construction of Φ_r in terms of the connection forms of M and dual one forms of the principal frame of Γ .

We describe a geometric construction for $\Phi := \Phi_{n-1}$, which avoids exterior algebra. This is of special interest in connection with conjecture (3). Let $GK := \sigma_{n-1}(\kappa)$ be the Gauss-Kronecker curvature of Γ . To motivate our approach, note that in \mathbb{R}^n

$$\nu^*(\operatorname{dvol}_{\mathbf{S}^{n-1}}) = GK\operatorname{dvol}_{\Gamma},$$

where dvol stands for volume form. So the Gauss-Kronecker curvature of Γ is the Jacobian of ν viewed as the Gauss map $\Gamma \to \mathbf{S}^{n-1}$. Hence we may set $\Phi := \operatorname{dvol}_{\mathbf{S}^{n-1}}$. To extend this concept to M, note that each tangent space of SM admits a decomposition [13, Sec. 1.3] into "horizontal" and "vertical" components given by

$$T_{(p,v)}SM = T_pM \oplus (v^{\perp}) \simeq T_pM \oplus T_vS_p,$$

where $v^{\perp} \subset T_pM$ is the subspace orthogonal to v, which may be identified with T_vS_p by parallel transport within T_pM . Let $\pi: T_{(p,v)}SM \to T_vS_p$ be projection onto the vertical component. Then we set

$$\Phi_{(p,v)} := \pi^*(\operatorname{dvol}_{S_p})_v.$$

To check this construction, let $e_i \in T_p\Gamma$ be an orthonormal set of principal directions with corresponding curvatures κ_i . Note that $d\pi = \pi$ and $d\overline{\nu}(e_i) = (e_i, \nabla_{e_i}\nu)$, where ∇ is the covariant derivative. Thus $d(\pi \circ \overline{\nu})(e_i) = \pi(e_i, \nabla_{e_i}\nu) = \nabla_{e_i}\nu = \kappa_i e_i$. So we have

$$(\overline{\nu}^* \Phi)_p(e_1, \dots, e_{n-1}) = (\pi \circ \overline{\nu})^* (\operatorname{dvol}_{S_p})_{\nu(p)}(e_1, \dots, e_{n-1})$$

= $(\operatorname{dvol}_{S_p})_{\nu(p)}(\kappa_1 e_1, \dots, \kappa_{n-1} e_{n-1}) = GK (\operatorname{dvol}_{\Gamma})_p(e_1, \dots, e_{n-1}).$

Hence $\overline{\nu}^*\Phi = GK \operatorname{dvol}_{\Gamma}$, as desired.

3. Proof of Theorem 1.1

Let ν be a unit normal vector field along Γ , and $u: M \to \mathbf{R}$ be the signed distance function of Γ with respect to ν . For $0 < \delta < \operatorname{reach}(\Gamma)$, let $U := u^{-1}([-\delta, \delta])$ be the tubular neighborhood of Γ of radius δ . Then $u \in \mathcal{C}^{1,1}(U)$ [9, Lem. 2.6], which means that u is $\mathcal{C}^{1,1}$ in a collection of local coordinate charts of M covering U. Fix $\delta < \min\{\operatorname{reach}(\Gamma)/2, \operatorname{reach}(\Gamma_i)/2\}$. We assume i is so large that $\Gamma_i \subset U$.

First assume that Γ_i do not intersect Γ . Let Ω_i be the region between Γ and Γ_i in U. Choose a unit normal vector field ν_i along Γ_i so that ν_i points into (away from) Ω_i , if ν points away from (into) Ω_i . Let u_i be the signed distance function of Γ_i with respect to ν_i . Note that $u_i \in \mathcal{C}^{1,1}(U)$, since $\delta < \operatorname{reach}(\Gamma_i)/2$. We say that $\Gamma_i \to \Gamma$ in $\mathcal{C}^{1,1}$ topology, provided that $u_i \to u$ in $\mathcal{C}^{1,1}(U)$, which means that $u_i \to u$ with respect to the $\mathcal{C}^{1,1}$ -norm in a collection of local charts covering U.

Lemma 3.1. $\Gamma_i \to \Gamma$ in $C^{1,1}$ topology.

Proof. By [9, Prop. 2.8], the Hessians of u and u_i are uniformly bounded almost everywhere on U. Hence, on U, the gradients ∇u and ∇u_i are uniformly Lipschitz. It only remains to check that $\nabla u_i \to \nabla u$ pointwise on U. For any point $p \in U \setminus \Gamma$ there exists a (geodesic) sphere $S \subset U$ of radius |u(p)| > 0 centered at p with $S \cap \Gamma = \{\overline{p}\}$. Similarly, assuming i is so large that $p \notin \Gamma_i$, there exists a sphere $S_i \subset U$ of radius $|u_i(p)| > 0$ centered at p with $S_i \cap \Gamma_i = \{\overline{p}_i\}$. Since $\Gamma_i \to \Gamma$ in Hausdorff distance, $u_i \to u$ in $C^0(U)$. Thus

$$\operatorname{dist}(\overline{p}_i, S) \leq \operatorname{dist}(S_i, S) \to 0,$$
 and $\operatorname{dist}(\overline{p}_i, \Gamma) \leq \operatorname{dist}(\Gamma_i, \Gamma) \to 0,$

where dist is the Riemannian distance in M. Thus any limit point of \overline{p}_i lies in $\Gamma \cap S = \{\overline{p}\}$, or $\overline{p}_i \to \overline{p}$. It follows that $\nabla u_i(p) \to \nabla u(p)$, since these gradients are unit tangent vectors at p to geodesic segments $p\overline{p}$ and $p\overline{p}_i$. Since the gradients are uniformly Lipschitz, $\nabla u_i(p) \to \nabla u(p)$ for all $p \in U$, which completes the proof.

The above lemma together with Kirszbraun's extension theorem leads to:

Lemma 3.2. There exists N > 0 such that, for i > N, ν_i extends to a uniformly Lipschitz unit normal vector field on Ω_i which coincides with ν on Γ .

Proof. Suppose first that U is parallelizable, so that there exists a smooth orthonormal frame field e_j on U. Then any unit vector in T_pU is identified with a point of \mathbf{S}^{n-1} by

$$v \mapsto (\langle v, e_1 \rangle, \dots, \langle v, e_n \rangle),$$

where $\langle \cdot, \cdot \rangle$ denotes the metric on M. In particular, the union of ν and ν_i yields a mapping $w_i \colon \partial \Omega_i \to \mathbf{S}^{n-1}$. By Lemma 3.1, w_i are uniformly Lipschitz, say with constant L. So by Kirszbraun's theorem, w_i admits an L-Lipschitz extension $\Omega_i \to \mathbf{R}^n$, which we again denote by w_i . For any point $p \in \Omega_i$ let $\overline{p} \in \Gamma$ be its nearest point. Then $\operatorname{dist}(p, \overline{p}) \leq \delta_i$, where δ_i denotes the maximum length in Ω_i of the geodesics orthogonal to Γ . Assume N is so large that $\delta_i < 1/L$. Then, by the triangle inequality,

$$|w_i(p)| \ge |w_i(\overline{p})| - |w_i(p) - w_i(\overline{p})| \ge 1 - L\delta_i > 0.$$

Hence $\nu_i := w_i/|w_i|$ yields the desired extension.

If U is not parallelizable, cover Γ by open topological balls $B_k \subset \Gamma$. Let $U_k \subset U$ be the cylindrical neighborhoods foliated by geodesics in U which are orthogonal to B_k . Then U_k admits a smooth orthonormal frame field. So, as discussed above, there exists

a unit normal vector field ν_i^k on $\Omega_i \cap U_k$ which is L-Lipschitz, and coincides with ν and ν_i on $\partial \Omega_i \cap U_k$. Let ϕ_k be a partition of unity on U subordinate to $\{U_k\}$, and set

$$\nu_i := \sum_k \phi_k \nu_i^k / \Big| \sum_k \phi_k \nu_i^k \Big|.$$

We claim that for N sufficiently large, $\sum_k \phi_k \nu_i^k \neq 0$ and thus ν_i is well-defined. Indeed for any point $p \in \Omega_i \cap U_k$, we have $\overline{p} \in \Gamma \cap U_k$. Thus

$$|\nu_i^k(p) - \nu(\overline{p})| = |\nu_i^k(p) - \nu_i^k(\overline{p})| \le L\delta_i.$$

Consequently, if $p \in \Omega_i \cap U_k \cap U_\ell$, then

$$|\nu_i^k(p) - \nu_i^\ell(p)| \le |\nu_i^k(p) - \nu(\overline{p})| + |\nu(\overline{p}) - \nu_i^\ell(\overline{p})| \le 2L\delta_i.$$

So if N is sufficiently large, $|\nu_i^k(p) - \nu_i^\ell(p)| \le 1$; in particular, these vectors all lie in an open hemisphere. Hence $\sum \phi_k \nu_i^k \ne 0$, which ensures that ν_i is the desired extension. \square

Let ν_i be the extension given by Lemma 3.2. Then by Lemma 2.1 and Stokes theorem, there exists a constant C independent of i such that

$$(4) \quad \left| \mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma) \right| = \left| \int_{\partial \Omega_i} \overline{\nu}_i^*(\Phi_r) \right| = \left| \int_{\Omega_i} d(\overline{\nu}_i^*(\Phi_r)) \right| = \left| \int_{\Omega_i} \overline{\nu}_i^*(d\Phi_r) \right| \le C|\Omega_i|,$$

since ν_i are uniformly Lipschitz, and so the pullbacks $\overline{\nu}_i^*$ are uniformly bounded. Thus $|\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma)| \to 0$, which concludes the proof in the case where $\Gamma_i \cap \Gamma = \emptyset$.

To prove the general case, let $\Gamma' \subset U$ be a hypersurface parallel to Γ , i.e., a level set of u different from Γ . Let Ω' be the region between Γ' and Γ . Then Γ_i will be disjoint from Γ' for i sufficiently large. Let Ω'_i be the region between Γ' and Γ_i . Since Γ , Γ' , and Γ_i all have uniformly positive reach, the same argument for (4) shows that

$$|\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma')| \le C|\Omega'_i|, \quad \text{and} \quad |\mathcal{M}_r(\Gamma') - \mathcal{M}_r(\Gamma)| \le C|\Omega'|,$$

for some constant C. Thus, by the triangle inequality,

$$\lim_{i \to \infty} |\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma)| \le \lim_{i \to \infty} C(|\Omega_i'| + |\Omega'|) \le 2C|\Omega'|.$$

As $\Gamma' \to \Gamma$, we have $|\Omega'| \to 0$, which completes the proof.

4. Proof of Theorem 1.2

Let M be a Cartan-Hadamard manifold and Γ be a convex hypersurface in M. Recall that Γ^{ε} denote the outer parallel hypersurfaces of Γ at distance $\varepsilon \geq 0$. Let Ω_{ε} be the region between Γ^{ε} and Γ , and K_M denote the sectional curvature of M.

Lemma 4.1. Let $C := \sup_{\Omega_{\varepsilon}} |K_M|$. Then

$$|\mathcal{M}_r(\Gamma^{\varepsilon}) - \mathcal{M}_r(\Gamma)| \le ((r+1)\mathcal{M}_{r+1}(\Gamma^{\varepsilon}) + C\mathcal{M}_{r-1}(\Gamma^{\varepsilon}))\varepsilon.$$

Proof. Let u be the distance function of Γ . Then $|\nabla u| = 1$ on the exterior region of Γ , and [11, Thm. 3.1] quickly yields

$$\mathcal{M}_r(\Gamma^{\varepsilon}) - \mathcal{M}_r(\Gamma) \le (r+1) \int_{\Omega_{\varepsilon}} \sigma_{r+1}(\kappa^u) + C \int_{\Omega_{\varepsilon}} \sigma_{r-1}(\kappa^u),$$

where $\kappa^u = (\kappa_1^u, \dots, \kappa_{n-1}^u)$ refers to the principal curvatures of the level sets of u. More precisely, we apply [11, Thm. 3.1] to parallel hypersurfaces Γ^{δ} for $0 < \delta < \varepsilon$ and take the limit as $\delta \to 0$ to obtain the above inequality. By the coarea formula

$$\int_{\Omega_{\varepsilon}} \sigma_{r+1}(\kappa^u) = \int_{0 < t < \varepsilon} \mathcal{M}_{r+1}(\Gamma^t) \le \varepsilon \mathcal{M}_{r+1}(\Gamma^{\varepsilon}),$$

where the last inequality is due to the monotonicity property (2). Similarly, $\int_{\Omega_{\varepsilon}} \sigma_{r-1}(\kappa^u) \leq \varepsilon \mathcal{M}_{r-1}(\Gamma^{\varepsilon})$, which completes the proof.

Suppose there exists a sequence of convex hypersurfaces $\Gamma_i \subset M$ such that $\Gamma_i \to \Gamma$ with respect to Hausdorff distance. By the triangle inequality,

$$|\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma)| \le |\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma_i^{\varepsilon})| + |\mathcal{M}_r(\Gamma_i^{\varepsilon}) - \mathcal{M}_r(\Gamma^{\varepsilon})| + |\mathcal{M}_r(\Gamma^{\varepsilon}) - \mathcal{M}_r(\Gamma)|.$$

As $i \to \infty$, the middle term on the right hand side vanishes by Theorem 1.1. To bound the first term, let $B \subset M$ be a closed ball which contains Γ in its interior, and set $C := \sup_B |K_M|$. For i sufficiently large and small ε , we have Γ_i , $\Gamma_i^{\varepsilon} \subset B$; therefore, Lemma 4.1 yields

$$|\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma_i^{\varepsilon})| \le ((r+1)\mathcal{M}_{r+1}(\Gamma_i^{\varepsilon}) + C\mathcal{M}_{r-1}(\Gamma_i^{\varepsilon}))\varepsilon.$$

But $\mathcal{M}_{r+1}(\Gamma_i^{\varepsilon}) \to \mathcal{M}_{r+1}(\Gamma^{\varepsilon})$ and $\mathcal{M}_{r-1}(\Gamma_i^{\varepsilon}) \to \mathcal{M}_{r-1}(\Gamma^{\varepsilon})$ by Theorem 1.1, since these hypersurfaces have uniformly positive reach. Thus

$$\lim_{i\to\infty} |\mathcal{M}_r(\Gamma_i) - \mathcal{M}_r(\Gamma)| \le ((r+1)\mathcal{M}_{r+1}(\Gamma^{\varepsilon}) + C\mathcal{M}_{r-1}(\Gamma^{\varepsilon}))\varepsilon + |\mathcal{M}_r(\Gamma^{\varepsilon}) - \mathcal{M}_r(\Gamma)|.$$

Letting $\varepsilon \to 0$ and recalling (1) completes the proof.

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