h-PRINCIPLES FOR CURVES AND KNOTS OF CONSTANT TORSION

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Abstract. We prove that curves of constant torsion satisfy the $\mathcal{C}^1$-dense h-principle in the space of immersed curves in Euclidean space. In particular, there exists a knot of constant torsion in each isotopy class. Our methods, which involve convex integration and degree theory, quickly establish these results for curves of constant curvature as well.

1. Introduction

Curves of constant torsion, which occur naturally as elastic rods, have long been studied [1–3, 12, 13, 16, 18, 22, 23], and some knotted examples have been found by various means. Here we construct knots of constant torsion in every isotopy class by adapting the convex integration [5, 11, 20] techniques developed for curves of constant curvature [7, 10]. To state our main result, let Γ be an interval $[a, b] \subset \mathbf{R}$ or topological circle $\mathbf{R}/((b-a)\mathbf{Z})$, and $\mathcal{C}^\alpha(\Gamma, \mathbf{R}^3)$ be the space of $\mathcal{C}^\alpha$ curves $f : \Gamma \to \mathbf{R}^3$ with its standard norm $|\cdot|_\alpha$. Let $\mathrm{Imm}^{\alpha\geqslant 1}(\Gamma, \mathbf{R}^3) \subset \mathcal{C}^\alpha(\Gamma, \mathbf{R}^3)$ consist of curves with speed $|f'| \neq 0$. If $|f'| = 1$, the *curvature* and *torsion* of f are given by $\kappa := |f''|$ and $\tau := \det(f', f'', f''')/\kappa^2$ respectively.

Theorem 1.1. *Let $f \in \mathrm{Imm}^{\alpha\geqslant 4}(\Gamma, \mathbf{R}^3)$ be a curve with κ, $\tau > 0$, and $p_i \in \Gamma$ be a finite collection of points. Then for any $\varepsilon > 0$ there exists a curve $\widetilde{f} \in \mathrm{Imm}^{\alpha-1}(\Gamma, \mathbf{R}^3)$ with $\widetilde{\kappa} > 0$ and $\widetilde{\tau} = constant$ such that $|\widetilde{f} - f|_1 \leqslant \varepsilon$ and $\widetilde{f}$ is tangent to f at p_i.*

If ε is sufficiently small, then $h_t := (1-t)f + t\widetilde{f}$, $t \in [0, 1]$, is a homotopy in $\mathrm{Imm}^{\alpha-1}(\Gamma, \mathbf{R}^3)$. In the terminology of Gromov or Eliashberg [4, 11], this constitutes a $\mathcal{C}^1$-dense h-principle for curves of constant torsion. Let $\mathrm{Emb}^\alpha(\Gamma, \mathbf{R}^3) \subset \mathrm{Imm}^\alpha(\Gamma, \mathbf{R}^3)$ be the space of injective curves, which are called *knots* when Γ is a circle. If $f \in \mathrm{Emb}^1(\Gamma, \mathbf{R}^3)$ and ε is sufficiently small, then h_t is an isotopy. Thus, since curves in $\mathrm{Emb}^\infty(\Gamma, \mathbf{R}^3)$ with κ, $\tau > 0$ are dense in $\mathrm{Emb}^1(\Gamma, \mathbf{R}^3)$, we obtain:

Corollary 1.2. *Every knot $f \in \mathrm{Emb}^1(\Gamma, \mathbf{R}^3)$ is isotopic in $\mathrm{Emb}^1(\Gamma, \mathbf{R}^3)$ to a knot $\widetilde{f} \in \mathrm{Emb}^\infty(\Gamma, \mathbf{R}^3)$ with $\widetilde{\kappa} > 0$ and $\widetilde{\tau} = constant$.*

Analogous results for curvature were established in [7], see also [10, 21] for related h-principles, and [15, 17] for earlier constructions. As in [7], we prove Theorem 1.1 by reducing it to a problem for spherical curves (Propositions 3.1 and 3.2). More explicitly, assuming $|f'| = 1$, we deform the *tantrix* $T := f'$ of f to a longer spherical curve $\widetilde{T}$ with $|\widetilde{T} - T|_0 \leqslant \varepsilon$, and then integrate $\widetilde{T}$ to obtain $\widetilde{f}$. For $\widetilde{\tau}$ to be constant, the product $\widetilde{k}\widetilde{v}^2$ must be constant (Lemma 2.1), where $\widetilde{k}$ is the geodesic curvature

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and $\widetilde{v}$ is the speed of $\widetilde{T}$. Furthermore, for $\widetilde{f}$ to be tangent to f at p_i we need to have $\int \widetilde{T} = \int T$ on every interval between p_i. We will show that these requirements can be met via basic convex geometry together with degree theory (Lemma 4.1), which makes the arguments significantly shorter than those in [7], although less explicit.

Constructing submanifolds with prescribed tangential directions has been a major theme in h-principle theory, e.g., see [9] and references therein. In particular see [6, 8] for more results and applications of curves with prescribed tantrices.

Note 1.3. Our methods also establish the analogue of Theorem 1.1 for curvature, with obvious simplifications since $\widetilde{T}$ would only need to have constant speed. In particular Lemma 2.1 below is not needed. Furthermore, in Proposition 3.1 we may replace the condition $\widetilde{\tau} = c$ with $\widetilde{\kappa} = c$, and in Proposition 3.2 replace $\widetilde{k}\widetilde{v}^2 = c$ with $\widetilde{v} = c$. The proofs will then proceed along the same lines, with only some abbreviations.

2. Reparametrization of the Tantrix

We begin by constructing constant torsion curves with a prescribed tantrix image. Set $I := [a, b]$, and $|I| := b - a$. Let $f \in \mathrm{Imm}^3(I, \mathbf{R}^3)$ be a curve with $|f'| = 1$, and set $v := |T'| = \kappa$. If $v \neq 0$, then $T \in \mathrm{Imm}^2(I, \mathbf{S}^2)$, and $N := T'/v$, $B := T \times N$ generate the Frenet frame (T, N, B). Then we may compute that

$$\tau = \langle N', B\rangle = \frac{\langle vT'' - v'T', B\rangle}{v^2} = \frac{\langle T'' - v'N, B\rangle}{v} = \frac{\langle T'', B\rangle}{v} = kv^2,$$

where $k := \langle T'', B\rangle / v^3$ is the geodesic curvature of T. We say $\widetilde{T}$ is a *reparametrization* of T if $\widetilde{T} = T \circ \varphi$ for an increasing diffeomorphism $\varphi \colon I \to I$. Standard ODE theory yields:

Lemma 2.1. *Let $T \in \mathrm{Imm}^{\alpha \geqslant 3}(I, \mathbf{S}^2)$ be a curve with $k > 0$. Then T admits a unique reparametrization $\widetilde{T} = T \circ \varphi \in \mathrm{Imm}^{\alpha-1}(I, \mathbf{S}^2)$ such that $\widetilde{k}\,\widetilde{v}^2$ is constant.*

Proof. For $c > 0$, the equation $\widetilde{k}\,\widetilde{v}^2 = c$ may be rewritten as $(v \circ \varphi)\varphi' = \sqrt{c/(k \circ \varphi)}$ by the chain rule and invariance of geodesic curvature. Since $\alpha \geqslant 3$, v and k are Lipschitz, and may be extended to Lipschitz functions on $\mathbf{R}$ without loss of regularity. So we arrive at the initial value problem

$$\begin{cases} \varphi' = F_c(\varphi), \\ \varphi(a) = a; \end{cases} \qquad \text{where} \qquad F_c(\cdot) := \frac{\sqrt{c}}{v(\cdot)\sqrt{k(\cdot)}}.$$

Since $F_c \colon \mathbf{R} \to \mathbf{R}$ is Lipschitz, for every c there exits a unique solution $\varphi_c \colon I \to \mathbf{R}$ by Picard–Lindelöf theorem. Since F_c is $\mathcal{C}^{\alpha-2}$, φ_c is $\mathcal{C}^{\alpha-1}$, and since $\varphi_c' \neq 0$, φ_c is a diffeomorphism onto its image. Note that $c \mapsto \varphi_c(b)$ is a continuous monotonic function since φ_c depends continuously on c and F_c varies monotonically with c. Furthermore $\varphi_c(b)$ can be made arbitrarily small or large along with F_c. Hence $\varphi_{c_0}(b) = b$ for a unique c_0, which yields the desired reparametrization. □

Suppose now that $f \in \mathrm{Imm}^{\alpha \geqslant 4}(I, \mathbf{R}^3)$ is a curve with κ, $\tau > 0$ and tantrix T. Then $T \in \mathrm{Imm}^{\alpha-1}(I, \mathbf{S}^2)$ with geodesic curvature $k = \tau/\kappa^2 > 0$. So we may apply the above lemma to obtain the reparametrization $\widetilde{T} \in \mathrm{Imm}^{\alpha-2}(I, \mathbf{S}^2)$. Then

$$\widetilde{f}(t) := f(a) + \int_a^t \widetilde{T}(u)du \tag{1}$$

is a $\mathcal{C}^{\alpha-1}$ curve of constant torsion c with $\widetilde{T}(I) = T(I)$. Since $c = \widetilde{k}\widetilde{v}^2 = (k \circ \varphi)\widetilde{v}^2$, and $L := \mathrm{length}(T) = \mathrm{length}(\widetilde{T}) = \int_I \widetilde{v}$, we obtain the following estimate

$$\frac{L^2}{|I|^2}\min_I(k) \;\leqslant\; c = \left(\frac{L}{|I| \ \mathrm{ave}_I\left((k\circ\varphi)^{-1/2}\right)}\right)^2 \;\leqslant\; \frac{L^2}{|I|^2}\max_I(k), \tag{2}$$

where $\mathrm{ave}_I(\cdot) := \int_I(\cdot)/|I|$.

3. Reduction to Spherical Curves

Here we use tantrices to reduce Theorem 1.1 to a problem for spherical curves. First we show that Theorem 1.1 follows from a more geometric local result. We say that a constant c is *arbitrarily large* if it can be chosen from an interval $[a, \infty)$.

Proposition 3.1. *Let $f \in \mathrm{Imm}^{\alpha\geqslant 4}(I, \mathbf{R}^3)$ be a curve with κ, $\tau > 0$ and V be an open neighborhood of $T(I)$ in $\mathbf{S}^2$. Then there exists a unit-speed curve $\widetilde{f} \in \mathrm{Imm}^{\alpha-1}(I, \mathbf{R}^3)$ with $\widetilde{\kappa} > 0$ and $\widetilde{\tau} = c$, for c arbitrarily large, such that $\widetilde{T}(I) \subset V$, $f = \widetilde{f}$ on ∂I, and $T(U) = \widetilde{T}(\widetilde{U})$ for some open neighborhoods U, $\widetilde{U}$ of ∂I in I.*

Proposition 3.1 implies Theorem 1.1 as follows. Let I_i be a partition of Γ into intervals such that ∂I_i include the prescribed points p_j. Choose I_i so small that $T(I_i)$ lies in the interior V_i of a disk of radius $\varepsilon/2$ in $\mathbf{S}^2$. Applying Proposition 3.1 to $f_i := f|_{I_i}$, we obtain $\mathcal{C}^{\alpha-1}$ curves $\widetilde{f}_i$ with $\widetilde{\tau}_i = c_i$, $\widetilde{f}_i = f_i$ on ∂I_i, $\widetilde{T}_i(I_i) \subset V_i$, and $T_i(U_i) = \widetilde{T}_i(\widetilde{U}_i)$ for open neighborhoods $U_i, \widetilde{U}_i$ of ∂I_i in I_i. The last condition yields open neighborhoods W_i and $\widetilde{W}_i$ of ∂I_i in Γ such that $\widetilde{T}(\widetilde{W}_i) = T(W_i)$. Furthermore, since c_i are arbitrarily large, they may assume the same value c. Define $\widetilde{f} \colon \Gamma \to \mathbf{R}^3$ by setting $\widetilde{f} := \widetilde{f}_i$ on I_i. Then $\widetilde{f}$ is $\mathcal{C}^0$, since $\widetilde{f} = f$ on ∂I_i. Since $\widetilde{f}_i' = \widetilde{T}_i$ and $\widetilde{T}_i = T_i$ on ∂I_i, $\widetilde{f}$ is $\mathcal{C}^1$. So $\widetilde{T} = \widetilde{f}'$ is well-defined. Note that $\widetilde{T}|_{I_i} = \widetilde{T}_i = \widetilde{f}_i'$ is $\mathcal{C}^{\alpha-2}$ and, since $\widetilde{T}(\widetilde{W}_i) = T(W_i)$, $\widetilde{T}$ is a reparametrization of T with speed $\sqrt{c/k}$ near ∂I_i. So, by Lemma 2.1, $\widetilde{T}$ is $\mathcal{C}^{\alpha-2}$. Hence $\widetilde{f}$ is $\mathcal{C}^{\alpha-1}$. Thus $\widetilde{\tau}$ is well-defined and is equal to c. Finally since $\widetilde{T}(I_i) \subset V_i$, $|\widetilde{T} - T|_0 \leqslant \varepsilon$, which yields $|\widetilde{f} - f|_1 \leqslant \varepsilon$ as desired.

It remains then to prove Proposition 3.1, which we reduce in turn to a more basic result. Let $\mathrm{conv}(f)$ denote the convex hull of f. Note that $\mathrm{ave}_I(f)$ lies in the relative interior of $\mathrm{conv}(f)$ [7, Lem. 2.1]. We say that f is *nonflat* provided that $\mathrm{conv}(f)$ has interior points, or $\mathrm{int}(\mathrm{conv}(f)) \neq \emptyset$.

Proposition 3.2. *Let $T \in \mathrm{Imm}^{\alpha\geqslant 3}(I, \mathbf{S}^2)$ be a nonflat curve with $k > 0$, $V \subset \mathbf{S}^2$ be an open neighborhood of $T(I)$, and $x_0 \in \mathrm{int}(\mathrm{conv}(T))$. Then there exists a curve $\widetilde{T} \in \mathrm{Imm}^{\alpha-1}(I, \mathbf{S}^2)$ with $\widetilde{k}\widetilde{v}^2 = c$, for c arbitrarily large, $\widetilde{T}(I) \subset V$, $T(U) = \widetilde{T}(\widetilde{U})$ for some open neighborhoods U, $\widetilde{U}$ of ∂I in I, and $\mathrm{ave}_I(\widetilde{T}) = x_0$.*

Proposition 3.2 implies Proposition 3.1 as follows. After a perturbation of f on a compact set in the interior of I we may assume that T is nonflat. Let $x_0 := \mathrm{ave}_I(T)$, $\widetilde{T}$ be the corresponding curve given by Proposition 3.2, and $\widetilde{f}$ be given by (1). Then $\widetilde{f}' = \widetilde{T}$. So $|\widetilde{f}'| = 1$, $\widetilde{\kappa} = |\widetilde{T}'| > 0$, and $\widetilde{\tau} = \widetilde{k}\widetilde{v}^2 = c$ as desired. Finally, $\int_I T = |I|\,\mathrm{ave}_I(T) = |I|\,\mathrm{ave}_I(\widetilde{T}) = \int_I \widetilde{T}$ which ensures that $f = \widetilde{f}$ on ∂I and completes the argument.

4. Controlling the Average

Here we establish Proposition 3.2, which completes the proof of Theorem 1.1 as discussed above. First we record the following basic fact.

Lemma 4.1. *Let $B \subset \mathbf{R}^n$ be a ball of radius R centered at x_0, and $F : B \to \mathbf{R}^n$ be a continuous map. If $|F(x) - x| < R$ for all $x \in \partial B$, then $x_0 \in F(B)$.*

Proof. For $t \in [0,1]$, let $F_t(x) := (1-t)x + tF(x)$, and set $f_t := F_t|_{\partial B}$. Since $|F(x) - x| < R$, for $x \in \partial B$, $x_0 \notin f_t(\partial B)$. So $\widetilde{f}_t := f_t/|f_t| : \partial B \to \partial B$ is well-defined. Since $\widetilde{f}_0$ is the identity map, $\deg(\widetilde{f}_0) = 1$. Thus, since $\widetilde{f}_t$ is a homotopy, $\deg(\widetilde{f}_1) = 1$. But $\widetilde{f}_1 = F/|F|$. So if $x_0 \notin F(B)$, $\widetilde{f}_1$ may be extended to B, which implies that it is homotopic to a constant map; therefore $\deg(\widetilde{f}_1) = 0$, which is a contradiction. □

Now we prove Proposition 3.2. By Steinitz's theorem [19, Thm. 1.3.10], there is a minimal set of points $v_i \in T$, $i = 1, \dots, n \leqslant 6$, such that $x_0 \in \mathrm{int}(\mathrm{conv}(\{v_i\}))$. Let $B \subset \mathrm{int}(\mathrm{conv}(\{v_i\}))$ be a ball of radius R centered at x_0. Then, by Carathéodory's theorem [19, Thm. 1.1.4], for each $x \in B$ there are constants $\lambda_i(x) > 0$, with $\sum_i \lambda_i(x) = 1$, such that $x = \sum_i \lambda_i(x) v_i$. By a theorem of Kalman [14], we may assume that $\lambda_i : B \to \mathbf{R}$ are continuous. Then $\overline{\lambda} := \min_B\{\lambda_i\} > 0$.

Let $C_i \subset V$ be circles of radius $r < R/(4n)$ which are tangent to T at v_i, and lie on the convex side of T. Let $\widetilde{k}_C$ be the geodesic curvature of C_i, which depends only on r. Choose r so small that $\widetilde{k}_C \geqslant \max_I(k)$. Then, after a perturbation of T, we may assume that a neighborhood of v_i in T lies on C_i, see Figure 1. Let

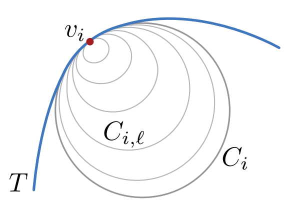

Figure 1.

$C_{i,\ell}$ be a continuous family of nested $\mathcal{C}^\alpha$ curves of length $0 < \ell < \mathrm{length}(C_i)$ and nondecreasing curvature which contain an open neighborhood of v_i in T and shrink to v_i as $\ell \to 0$. Then for any $\ell_i > 0$ we can construct a unique composite loop of length ℓ_i at v_i as follows. Let m be the largest integer with $m\ell_i \leqslant \mathrm{length}(C_i)$. Then go m times around C_i and once around $C_{i,\ell}$ for $\ell = \mathrm{length}(C_i) - m\ell_i$.

Set $L := \text{length}(T)$, and choose $\widetilde{L} > L$. For $x \in B$, let $\overline{T}_x \in \text{Imm}^\alpha(I, \mathbf{S}^2)$ be the constant speed curve of length $\widetilde{L}$ which traces the image of T plus loops of length

$$\ell_i(x) := \lambda_i(x)(\widetilde{L} - L)$$

at v_i. Then $x \mapsto \overline{T}_x$ is continuous with respect to the $\mathcal{C}^0$-norm on $\mathcal{C}^0(I, \mathbf{S}^2)$. Next let $\widetilde{T}_x$ be the reparametrizations of $\overline{T}_x$ given by Lemma 2.1, and note that $\overline{T}_x \mapsto \widetilde{T}_x$ is continuous with respect to the $\mathcal{C}^0$-norm. Thus the mapping

$$B \ni x \xmapsto{\;F\;} \text{ave}_I(\widetilde{T}_x) \in \mathbf{R}^3$$

is continuous. We may assume that $|I| = 1$. Then by (2) and since $\widetilde{k}_C \geqslant \max_I(k)$,

$$c = \left(\widetilde{L}/\,\text{ave}_I\left(\widetilde{k}_x^{-1/2}\right)\right)^2 \geqslant \widetilde{L}^2 \min_I(\widetilde{k}_x) = \widetilde{L}^2 \min_I(k).$$

So $c \to \infty$ as $\widetilde{L} \to \infty$. We will show that if $\widetilde{L}$ is sufficiently large, then $|F(x) - x| < R$, which will complete the proof by Lemma 4.1.

Let $\widetilde{T}_x^i$ be the part of $\widetilde{T}_x$ which forms the loop at v_i, and $I_x^i \subset I$ be the subinterval such that $\widetilde{T}_x^i = \widetilde{T}|_{I_x^i}$. Set $I_x' := I - \cup_i I_x^i$. Then

$$F(x) = \sum_i |I_x^i|\, \text{ave}_{I_x^i}(\widetilde{T}_x^i) + |I_x'|\, \text{ave}_{I_x'}(\widetilde{T}_x).$$

Since $\text{ave}_{I_x^i}(\widetilde{T}_x^i) \in \text{conv}(\widetilde{T}_x^i) \subset \text{conv}(D_i)$, where $D_i \subset \mathbf{S}^2$ is the small disk bounded by C_i, we have $|\text{ave}_{I_x^i}(\widetilde{T}_x^i) - v_i| \leqslant 2r < R/(2n)$. So subtracting $\sum_i |I_x^i| v_i$ from both sides of the above equation yields

$$\left| F(x) - \sum_i |I_x^i| v_i \right| < \frac{R}{2} + |I_x'|\big|\,\text{ave}_{I_x'}(\widetilde{T}_x)\big|. \tag{3}$$

Now let $\widetilde{L} \to \infty$. Note that $|I_x'| = L/\,\text{ave}_{I_x'}((c/\widetilde{k}_x)^{1/2})$, and $\min_{I_x'}(\widetilde{k}_x) = \min_I(k) > 0$, since $\widetilde{T}_x(I_x') = T(I)$. So $|I_x'| \to 0$. But, since $\text{ave}_{I_x'}(\widetilde{T}_x) \in \text{conv}(T)$, $|\text{ave}_{I_x'}(\widetilde{T}_x)|$ is bounded above. So the right hand side of (3) converges uniformly to $R/2$. Next note that

$$|I_x^i| = \frac{\ell_i(x)}{\sqrt{c}\,\text{ave}_{I_x^i}\left(\widetilde{k}_x^{-1/2}\right)} = \frac{\lambda_i(x)(\widetilde{L} - L)\,\text{ave}_I\left(\widetilde{k}_x^{-1/2}\right)}{\widetilde{L}\,\text{ave}_{I_x^i}\left(\widetilde{k}_x^{-1/2}\right)}.$$

We have $\ell_i \to \infty$ since $\lambda_i \geqslant \overline{\lambda} > 0$. So $\text{ave}_{I_x^i}(\widetilde{k}_x^{-1/2}) \to \widetilde{k}_C^{-1/2}$. Since $|I_x'| \to 0$, $\text{ave}_I(\widetilde{k}_x^{-1/2}) \to \widetilde{k}_C^{-1/2}$ as well. Thus $|I_x^i| \to \lambda_i(x)$. So the left hand side of (3) converges uniformly to $|F(x) - x|$, which completes the proof.

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