TOPOLOGY OF CLOSED ASYMPTOTIC CURVES ON NEGATIVELY CURVED SURFACES

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Abstract. Motivated by Nirenberg's problem on isometric rigidity of tight surfaces, we study planar projections of closed asymptotic curves Γ on negatively curved surfaces M in Euclidean 3-space. In particular, using Călugăreanu's theorem, we obtain a formula for the linking number $\mathrm{Lk}(\Gamma, n)$ of Γ with the normal n of M. It follows that when $\mathrm{Lk}(\Gamma, n) = 0$, Γ cannot have any locally star-shaped projections with vanishing crossing number, which extends observations of Kovaleva, Panov and Arnold. These results hold also for curves with nonvanishing torsion and their binormal vector field. Finally we construct an example where n is injective but $\mathrm{Lk}(\Gamma, n) \neq 0$.

1. Introduction

An *asymptotic curve* on a negatively curved surface M in Euclidean space $\mathbf{R}^3$ is an integral curve of the directions where the second fundamental form of M vanishes. These objects form characteristic curves of the hyperbolic PDE for isometric embeddings, and thus play a fundamental role in surface theory. In particular, a well-known problem of Nirenberg [20–22, 25, 29, 35, 37], on rigidity of tight surfaces [3, 7], is concerned with existence of closed asymptotic curves Γ when M forms the interior of an annular surface $\overline{M}$ bounded by convex curves in planes tangent to $\overline{M}$. In this setting the Gauss map $n\colon M \to \mathbf{S}^2$ is injective, see Note 1.5. Furthermore, the linking number $\mathrm{Lk}(\Gamma, n) = 0$ if M is embedded. Thus we consider:

Problem 1.1. *Let $M \subset \mathbf{R}^3$ be a negatively curved embedded surface with Gauss map n and $\Gamma \subset M$ be a closed asymptotic curve. Can $\mathrm{Lk}(\Gamma, n) = 0$ if n is injective?*

A negative answer will settle Nirenberg's problem in the embedded case (in general M may self-intersect [5, p. 68], so $\mathrm{Lk}(\Gamma, n)$ may not be well-defined). To study this problem we assume that M is $\mathcal{C}^3$, which ensures that Γ is $\mathcal{C}^2$, see Section 2. Let $u \in \mathbf{S}^2$ be a random direction, Γ_u be the projection of Γ into a plane orthogonal to u, $\mathrm{Cr}(\Gamma_u)$ be the the sum of signed crossings of Γ_u, $\#\big\{\langle u, n\rangle = 0\big\}$ be the number of zeros of $\langle u, n\rangle$, and τ_g be the geodesic torsion of Γ, which as we will show in Section 2 has a fixed sign. Our main results are as follows.

Date: Last revised on December 21, 2024.

1991 *Mathematics Subject Classification.* Primary 53A04, 53A05; Secondary 57K10, 35L10.

Key words and phrases. Rigidity of isometric embeddings, Tight surfaces, Characteristic curves, Hyperbolic PDE, Călugăreanu's formula, Self-linking number, Writhe, Torsion of closed curves.

The first-named author was supported by NSF grant DMS-2202337.

Theorem 1.2.

$$\mathrm{Lk}(\Gamma, n) = \mathrm{Cr}(\Gamma_u) + \frac{1}{2}\,\#\big\{\langle u, n\rangle = 0\big\}\,\mathrm{sign}(\tau_g). \tag{1}$$

This equation is a refinement of Călugăreanu's formula [9, 10, 32] for the self-linking number of curves, see Note 3.2. We say that Γ_u is *locally star-shaped* if its tangent lines do not cover the plane of Γ_u. Using the above formula we show:

Theorem 1.3. *If* $\mathrm{Cr}(\Gamma_u) = \mathrm{Lk}(\Gamma, n)$, *then* Γ_u *cannot be locally star-shaped.*

In particular, if $\mathrm{Lk}(\Gamma, n) = 0$, or M is isotopic to a planar annulus, then Γ_u cannot be star-shaped, as had been observed by Kovaleva [26]. Panov [30] and Arnold [4] reproved Kovaleva's result in the case where M forms a graph over the xy-plane. Petrunin [31, p. 31] also gave a proof in this case, see Note 3.1. Our proof of the above theorem leads to our next result. The *absolute rotation index* of a closed curve immersed in the plane is the absolute value of the degree of its Gauss map. An *inflection* is a point of vanishing curvature.

Theorem 1.4. *If* Γ_u *has no inflections, then it cannot be locally star-shaped; in particular, the absolute rotation index of* Γ_u *cannot be less than* 3.

Any space curve with nonvanishing torsion may be realized as an asymptotic curve on a negatively curved surface orthogonal to its binormal vector, see Note 2.1. Thus the above results hold for curves with nonvanishing torsion and their binormal. In particular, Theorem 1.4 implies that curves of type $(p, 1)$ on a torus of revolution cannot have nonvanishing torsion, as had been observed by Costa [33]. This result also generalizes the fact that a curve of nonvanishing torsion cannot project into a strictly convex planar curve, which has been observed in various contexts [6, 24]. See [8, 15–17] for other recent results on torsion.

When n is injective, Γ must have inflections, see Note 5.1. At these points the Frenet frame of Γ is not well-defined, but the Darboux frame of Γ with respect to M will be useful, as we describe in Section 2. Applying Călugăreanu's formula to the Darboux frame will yield Theorem 1.2, and Theorems 1.3 and 1.4 follow quickly in Section 3. In Section 4 we construct an example where $\mathrm{Lk}(\Gamma, n) \neq 0$ but n is injective, which complements constructions by Kovaleva [27] and Arnold [4] where $\mathrm{Lk}(\Gamma, n) = 0$ but n is noninjective. This example also contradicts a claim of Kovaleva [27], see Note 5.5. Other observations concerning Problem 1.1 will be discussed in Section 5.

Note 1.5. To see that the Gauss map n in Nirenberg's problem is injective on M note that since M has curvature $K < 0$ it must lie within the convex hull of $\partial\overline{M}$. Thus the inward normal of $\partial\overline{M}$ in $\overline{M}$ points to the interior of the convex bodies bounded by $\partial\overline{M}$. So the geodesic curvature of each component of $\partial\overline{M}$ is the same as the curvature of that component in the plane where it lies. Hence, by the convexity assumption, the total geodesic curvature of each component of $\partial\overline{M}$ is 2π. The Gauss-Bonnet theorem then yields that $\int_{\overline{M}} K = -4\pi$. But n is onto, since it maps each boundary component of $\overline{M}$ to a point, and is locally injective on M since $K \neq 0$. Hence n must be injective on M.

2. Preliminaries

2.1. **Regularity and simplicity of Γ.** Throughout this work we assume that $M \subset \mathbf{R}^3$ is a $\mathcal{C}^{k\geq 3}$ embedded orientable surface with Gauss curvature $K<0$, Gauss map n, second fundamental form $\mathrm{I\!I}(X,Y) := \langle dn(X), Y\rangle$, and closed asymptotic curve $\Gamma\colon \mathbf{S}^1 \simeq \mathbf{R}/(2\pi\mathbf{Z}) \to M$. We will write Γ to refer both to the mapping and its image $\Gamma(\mathbf{S}^1)$, and will write $n(t)$ to mean $n(\Gamma(t))$. We assume that the speed $|\Gamma'|=1$, after a rescaling, and set $T := \Gamma'$. Then

$$\mathrm{I\!I}(T,T)=0.$$

This equation may be written as $\ell_{11}(du_1/du_2)^2 + 2\ell_{12}(du_1/du_2) + \ell_{22} = 0$ in local coordinates where (u_1,u_2) represent Γ and ℓ_{ij} are the coefficients of $\mathrm{I\!I}$. Thus Γ is $\mathcal{C}^{k-1}$ since ℓ_{ij} are $\mathcal{C}^{k-2}$. It is easy to see that one can distinguish precisely two fields of asymptotic lines on M [29, p. 184]. Thus, by the uniqueness of solutions to ODE, Γ cannot self-intersect in M. Since M is embedded in $\mathbf{R}^3$, it follows that Γ is simple.

2.2. **The Darboux frame.** Set $n^\perp := n \times T$. Then $(T, n^\perp, n)$ forms the *Darboux frame* of Γ with respect to M. Since M is $\mathcal{C}^3$ and Γ is $\mathcal{C}^2$, $(T, n^\perp, n)$ is $\mathcal{C}^1$, and we have

$$T' = \kappa_g n^\perp, \qquad (n^\perp)' = -\kappa_g T + \tau_g n, \qquad n' = -\tau_g n^\perp, \tag{2}$$

where $\kappa_g := \langle T', n^\perp\rangle$ and $\tau_g := \langle (n^\perp)', n\rangle$ are the *geodesic curvature* and *geodesic torsion* of Γ respectively. Let $\kappa := |T'|$ be the curvature of Γ. Then $|\kappa_g| = \kappa$. So there is no distinction between inflections of Γ with respect to κ or κ_g. If $\kappa \neq 0$ at any point, then the principal normal $N := T'/\kappa$ and the binormal $B := T \times N$ of Γ generate the *Frenet frame* (T,N,B). We have $(N,B) = \pm(n^\perp, n)$ depending on the sign of κ_g. So when $\kappa \neq 0$ the Darboux and Frenet frames coincide up to reflections. In particular, when $k \geq 4$, or Γ is $\mathcal{C}^3$, the torsion $\tau := \langle N', B\rangle$ of Γ coincides with τ_g. Note that $|\tau_g| = |n'| = |dn(T)|$, and dn is nondegenerate since $K \neq 0$. Thus

$$\tau_g \neq 0. \tag{3}$$

This observation also follows from the Beltrami-Enneper theorem [36, p. 200] [23, p. 609].

2.3. **Crossings.** For any simple closed immersion $\Gamma\colon \mathbf{S}^1 \to \mathbf{R}^3$ and direction $u \in \mathbf{S}^2$, let Γ_u denote the projection of Γ into a plane Π orthogonal to u. For almost every u, or *random direction*, Γ_u is in *general position*. More explicitly, there are only finitely many points $p_i \in \Pi$, called *crossings*, such that $\Gamma_u^{-1}(p_i)$ consists of more than one point. Furthermore all p_i will be transversal double points, i.e., $\Gamma_u^{-1}(p_i) = \{t_i^+, t_i^-\}$ with $\Gamma_u'(t_i^+) \times \Gamma_u'(t_i^+) \neq 0$. We assume that $\langle \Gamma(t_i^+), u\rangle > \langle \Gamma(t_i^-), u\rangle$. Then $\mathrm{sign}(p_i)$ is defined as the sign of $\langle \Gamma_u(t_i^+) \times \Gamma_u(t_i^-), u\rangle$, i.e., 1 or -1 depending on whether $\langle \Gamma_u(t_i^+) \times \Gamma_u(t_i^-), u\rangle > 0$ or < 0 respectively. Then the *crossing number* of Γ_u is given by $\mathrm{Cr}(\Gamma_u) := \sum \mathrm{sign}(p_i)$.

For any pair of disjoint immersions Γ^1, $\Gamma^2\colon \mathbf{S}^1 \to \mathbf{R}^3$, the *crossing number* $\mathrm{Cr}(\Gamma_u^1, \Gamma_u^2)$ is defined similarly. Again, assuming u is a random direction or Γ^1 and Γ^2 are in general position, then $\Gamma_u^1 \cap \Gamma_u^2$ consists of only a finite number of points

p_i, and for each p_i there will be exactly one pair of points t_i^1, $t_i^2 \in \mathbf{S}^1$ such that $\Gamma_u^1(t_i^1) = \Gamma_u^2(t_i^2) = p_i$. Then $\operatorname{sign}(p_i)$ is defined as the sign of $\langle (\Gamma_u^1)'(t_i^1) \times (\Gamma_u^2)'(t_i^2), u\rangle$ or $\langle (\Gamma_u^2)'(t_i^2) \times \langle (\Gamma_u^1)'(t_i^1), u\rangle$ if $\langle \Gamma^1(t_i^1), u\rangle > \langle \Gamma^2(t_i^2), u\rangle$ or $\langle \Gamma^1(t_i^1), u\rangle < \langle \Gamma^2(t_i^2), u\rangle$ respectively. Finally we set $\operatorname{Cr}(\Gamma_u^1, \Gamma_u^2) := \sum \operatorname{sign}(p_i)$.

Let v be a unit normal vector field along Γ. The pair (Γ, v) is called a *ribbon* based on Γ. Let $\epsilon > 0$ be so small that the perturbation $\Gamma + \epsilon v$ is disjoint from Γ. The crossings $\Gamma_u \cap (\Gamma + \epsilon v)_u$ fall into two categories [11]: a crossing formed at $t \in \mathbf{S}^1$ is *local* if $\Gamma_u(t) = (\Gamma(t) + \epsilon v(t))_u$, or $v(t) = \pm u$; otherwise, it is *nonlocal*. Nonlocal crossings converge in pairs to self-crossings of Γ_u as $\epsilon \to 0$. Thus

$$\operatorname{Cr}\big(\Gamma_u, (\Gamma + \epsilon v)_u\big) = 2\operatorname{Cr}(\Gamma_u) + \operatorname{Cr}_{local}\big(\Gamma_u, (\Gamma + \epsilon v)_u\big), \tag{4}$$

where $\operatorname{Cr}_{local}$ is the signed sum of local crossings.

2.4. **Călugăreanu's formula.** For a ribbon (Γ, v), Călugăreanu's formula [1, 11, 13, 28] states that

$$\operatorname{Lk}(\Gamma, v) = \operatorname{Wr}(\Gamma) + \operatorname{Tw}(\Gamma, v),$$

where Lk, Wr and Tw stand for the linking number, writhe and twist respectively. The linking number is defined as $\operatorname{Cr}\big(\Gamma_u, (\Gamma + \epsilon v)_u\big)/2$. So by (4)

$$\operatorname{Lk}(\Gamma, v) = \operatorname{Cr}(\Gamma_u) + \frac{1}{2}\operatorname{Cr}_{local}\big(\Gamma_u, (\Gamma + \epsilon v)_u\big). \tag{5}$$

The *twist* of (Γ, v) is given by $\int_\Gamma \langle (v^\perp)', v\rangle/(2\pi)$ where $v^\perp := v \times T$. Let $\theta_v \colon \mathbf{S}^1 \to \mathbf{R}$ be the continuous function such that $v(t) = \cos(\theta_v(t))n^\perp(t) + \sin(\theta_v(t))n(t)$, and $\operatorname{Rot}(v, n) := (\theta_v(L) - \theta_v(0))/(2\pi)$ denote the *total rotation* of v with respect to n. A computation using (2) yields that

$$\operatorname{Tw}(\Gamma, v) = \frac{1}{2\pi}\int_\Gamma \tau_g + \operatorname{Rot}(v, n).$$

Finally, the *writhe* of Γ is the average of the crossing numbers of Γ_u,

$$\operatorname{Wr}(\Gamma) := \frac{1}{4\pi}\int_{u\in\mathbf{S}^2} \operatorname{Cr}(\Gamma_u). \tag{6}$$

Note that $\operatorname{Wr}(\Gamma) = \operatorname{Lk}(\Gamma, n) - \operatorname{Tw}(\Gamma, n) = \operatorname{Lk}(\Gamma, n) - \int_\Gamma \tau_g/(2\pi)$. Thus we conclude that

$$\operatorname{Lk}(\Gamma, v) = \operatorname{Lk}(\Gamma, n) + \operatorname{Rot}(v, n). \tag{7}$$

Note 2.1. A curve Γ in $\mathbf{R}^3$ which admits a family of nonstationary osculating planes, i.e., an orthonormal frame satisfying (2) and (3), has been called a *rotating curve* by Arnold [4]. These objects, which generalize curves of nonvanishing torsion, were also described by Fenchel [12]. If Γ is a rotating curve, then a routine computation shows that $\Gamma(t) + sv(t)$ generates a negatively curved surface, for $-\epsilon < s < \epsilon$, which contains Γ as an asymptotic curve. More specifically, $K(t, 0) = -\tau_g^2(t)$. Thus rotating curves are precisely asymptotic curves of negatively curved surfaces.

3. Proofs of the Main Results

3.1. **Proof of Theorem 1.2.** Let $u \in \mathbf{S}^2$ be a direction which is not parallel to any tangent line of Γ, and set $u^\perp := u \times T/|u \times T|$. By (7)

$$\mathrm{Lk}(\Gamma, n) = \mathrm{Lk}(\Gamma, u^\perp) - \mathrm{Rot}(u^\perp, n).$$

Note that $(\Gamma + \epsilon u^\perp)_u$ is locally disjoint from Γ_u. Thus (5) yields that

$$\mathrm{Lk}(\Gamma, u^\perp) = \mathrm{Cr}(\Gamma_u).$$

It remains then to compute $\mathrm{Rot}(u^\perp, n)$. To this end consider the mapping $\nu\colon \mathbf{S}^1 \to \mathbf{S}^1$ given by $\nu(t) := e^{i\theta(t)}$, where $\theta := \theta_{u^\perp}$ is as defined above, i.e., $u^\perp(t) = \cos(\theta(t))n^\perp + \sin(\theta(t))n$. Then $\mathrm{Rot}(u^\perp, n) = \deg(\nu)$. To compute $\deg(\nu)$ assume that $\mathbf{S}^1$ is oriented counterclockwise. So $it_0 \in T_{t_0}\mathbf{S}^1$ has positive orientation. We have $d\nu_{t_0}(it_0) = i\nu(t_0)\theta'(t_0) \in T_{\nu(t_0)}\mathbf{S}^1$. To find θ' note that

$$\cot(\theta) = \frac{\langle u^\perp, n^\perp\rangle}{\langle u^\perp, n\rangle} = \frac{\langle u \times T, n^\perp\rangle}{\langle u \times T, n\rangle} = \frac{\langle u, T \times n^\perp\rangle}{\langle u, T \times n\rangle} = -\frac{\langle u, n\rangle}{\langle u, n^\perp\rangle}.$$

Differentiating the far sides of this expression using (2), we obtain

$$\frac{\theta'}{\sin^2(\theta)} = -\tau_g - \frac{\langle u, n\rangle\langle u, -\kappa_g T + \tau_g n^\perp\rangle}{\langle u, n^\perp\rangle^2}.$$

Suppose $\nu(t_0) = (0, \pm 1)$, or $\theta(t_0) = m\pi + \pi/2$, for $m \in \mathbf{Z}$. Then $u^\perp(t_0) = n(t_0)$, which yields $\langle u, n(t_0)\rangle = 0$. It follows that $\theta'(t_0) = -\tau_g(t_0)$. So $d\nu_{t_0}(it_0) = -i\nu(t_0)\tau_g(t_0)$, which does not vanish by (3). Thus $(0, \pm 1)$ are regular values of ν, and $d\nu_{t_0}$ preserves orientation if and only if $\tau_g(t_0) < 0$. Hence, since τ_g has constant sign, $-\mathrm{sign}(\tau_g)\deg(\nu) = \#\{\nu^{-1}(0, \pm 1)\}/2 = \#\big\{\langle u, n\rangle = 0\big\}/2$. So

$$\mathrm{Rot}(u^\perp, n) = -\frac{1}{2}\,\#\big\{\langle u, n\rangle = 0\big\}\,\mathrm{sign}(\tau_g)$$

which completes the proof.

3.2. **Proof of Theorem 1.3.** If $\mathrm{Cr}(\Gamma_u) = \mathrm{Lk}(\Gamma, n)$ then $\langle u, n\rangle \neq 0$ by (1). So M is locally a graph over a plane orthogonal to u, which we may identify with the xy-plane. After an affine transformation, given by $(x, y, z) \mapsto (x, y, \lambda z)$, we may assume that M is arbitrarily close to the xy-plane, since affine transformations preserve asymptotic curves. Suppose, towards a contradiction, that Γ_u is locally star-shaped with respect to the origin, i.e., tangent lines of Γ_u do not pass through o. Then $\langle \Gamma_u, u^\perp\rangle \neq 0$. Let $h := \langle \Gamma, n\rangle$. Then $h' = -\tau_g\langle \Gamma, n^\perp\rangle$. But $\langle \Gamma, n^\perp\rangle \to \langle \Gamma_u, u^\perp\rangle$ as $\lambda \to 0$. So $h' \neq 0$, for λ small, which is impossible since Γ is closed.

3.3. **Proof of Theorem 1.4.** We may again assume that $u = (0, 0, 1)$ and Γ_u lies in the xy-plane. When the curvature of Γ_u does not vanish, the curvature κ of Γ does not vanish either, and the principal normal $N \neq \pm u$. But if κ does not vanish, then $n^\perp = \pm N$, as discussed in the last section. So $n^\perp \neq \pm u$. Then, after the rescaling $(x, y, z) \mapsto (x, y, \lambda z)$, we may assume that $|\langle T, u\rangle|$ and $|\langle n^\perp, u\rangle|$ are arbitrarily small, which implies that n is almost parallel to u or $-u$. In particular $\langle n, u\rangle \neq 0$. Thus

M is locally a graph over the xy-plane. Now if $\tau_g \neq 0$, then we again arrive at a contradiction, as shown in the proof of Theorem 1.3.

Note 3.1. An alternative argument for finishing the proofs of Theorems 1.3 and 1.4, once we know that M is locally a graph over the xy-plane, can be given following Petrunin [31, p. 31]. Let $u = (0,0,1)$ and set $h := \langle \Gamma, n\rangle / \langle u, n\rangle$, i.e., the height over the origin of the tangent planes of M along Γ. We compute that

$$-h'\langle u,n\rangle^2 = \tau_g\big(\langle \Gamma, n^\perp\rangle\langle u,n\rangle - \langle \Gamma, n\rangle\langle u, n^\perp\rangle\big) = \tau_g\langle \Gamma, T\times u\rangle = \tau_g|T\times u|\langle \Gamma, u^\perp\rangle.$$

If Γ_u is locally star-shaped with respect to the origin, then $\langle \Gamma, u^\perp\rangle = \langle \Gamma_u, u^\perp\rangle \neq 0$. So $h' \neq 0$, which is again a contradiction.

Note 3.2. As we mentioned in Section 2, if Γ has no inflections, then $n = \pm B$ and $\tau_g = \tau$. So $\mathrm{Lk}(\Gamma, n) = \mathrm{Lk}(\Gamma, B) = \mathrm{Lk}(\Gamma, N)$ which is known as the self-linking number of Γ and is denoted by $\mathrm{SL}(\Gamma)$ [32,34]. Thus (1) yields that

$$\mathrm{SL}(\Gamma) = \mathrm{Cr}(\Gamma_u) + \frac{1}{2}\#\big\{\langle u, B\rangle = 0\big\}\,\mathrm{sign}(\tau). \tag{8}$$

By Crofton's formula,

$$\int_{u\in\mathbf{S}^2} \#\big\{\langle u,B\rangle = 0\big\}\,\mathrm{sign}(\tau) = 2\,\mathrm{Length}(B)\,\mathrm{sign}(\tau) = 2\int_\Gamma \tau.$$

Thus (8) together with (6) yields $\mathrm{SL}(\Gamma) = \mathrm{Wr}(\Gamma) + \frac{1}{2\pi}\int_\Gamma \tau$, which is a special case of Călugăreanu's formula [32]. Hence (1) may be regarded as a generalization of Călugăreanu's formula for self-linking number of curves with nonvanishing curvature and torsion. For this class of curves (8) may be rewritten as

$$\mathrm{SL}(\Gamma) = \mathrm{Cr}(\Gamma_u) + \frac{1}{2}\mathrm{Inflection}(\Gamma_u),$$

where $\mathrm{Inflection}(\Gamma_u)$ denotes the number of inflections of Γ_u. Indeed $\Gamma_u(t)$ is an inflection if and only if the principal normal $N(t)$ projects into the tangent line of Γ_u at $\Gamma_u(t)$, or $N(t)$ lies in the plane spanned by $T(t)$ and u, which is the case if and only if $\langle u, B(t)\rangle = 0$.

4. Examples

Here we describe a pair of examples which illustrate some of the subtleties of Problem 1.1.

Example 4.1. For the sake of comparison we start with Kovaleva's example [26], which is given by the coordinate functions

$$\begin{aligned}
\Gamma_1(t) &:= (3+\sin(t))\cos(\sqrt{63/8}\cos(t)),\\
\Gamma_2(t) &:= (3+\sin(t))\sin(\sqrt{63/8}\cos(t)),\\
\Gamma_3(t) &:= \sin(2t) + 46\cos(t) - 27\cos^3(t) + 27/8\cos^5(t),
\end{aligned}$$

see Figure 1(a). We have provided a Mathematica notebook [18] where one can check that this curve admits a Darboux framing $(T, n^\perp, n)$ with nonvanishing torsion τ_g, as described in Note 2.1, and therefore is a rotating or asymptotic curve. Figure

(a) (b)

FIGURE 1.

1(b) shows that $n(\Gamma)$ is not injective. Furthermore, by (1) we have $\mathrm{Lk}(\Gamma, n) = 0$, since as Figure 1 shows, $\mathrm{Cr}(\Gamma_u) = 0 = \#\{\langle n, u\rangle = 0\}$ for $u = (0, 0, 1)$.

Aicardi [2] showed that curves of nonvanishing curvature and torsion can be constructed with any self-linking number. Thus there exists closed asymptotic curves with any self liking number, as we discussed in Note 2.1. The special feature of Kovaleva's example, and similar constructions by Arnold, however, is that the surface forms a graph, which makes them much more subtle.

Example 4.2. Next we construct a closed asymptotic curve with opposite properties, i.e., n injective but $\mathrm{Lk}(\Gamma, n) \neq 0$. Again the reader can verify our computations via the Mathematica notebook [18] that we have provided. This example is obtained by starting with an embedding $n\colon \mathbf{S}^1 \to \mathbf{S}^2$ given by

$$\begin{aligned} n_1(t) &:= (3 + \sin(t)) \cos(5/2 \cos(t)), \\ n_2(t) &:= (3 + \sin(t)) \sin(5/2 \cos(t)), \\ n_3(t) &:= (1 - n_1^2(t) - n_2^2(t))^{1/2}, \end{aligned}$$

see Figure 2(a). Set $\tau_g := |n'|$, and $T := -n \times n'/\tau_g$. Note that T traces the

(a) (b)

FIGURE 2.

center of oriented great circles tangent to n. We have constructed n so that these circles cover $\mathbf{S}^2$, i.e., n is not star-shaped. It follows that the origin is contained in the interior of the convex hull of T, see Figure 3. Thus there exists a $\mathcal{C}^\infty$ positive function $\rho\colon \mathbf{S}^1 \to \mathbf{R}$ with $\int \rho T = 0$, which can be constructed using methods of convex integration [14, Lem. 2.3] [19, p. 168]. Setting $\Gamma(t) := \int_0^t \rho(s) T(s) ds$, for an appropriate choice of ρ, yields the desired curve depicted in Figure 2(a). Here $\mathrm{Cr}(\Gamma_u) = 2$ whereas, $\#\{\langle n, u\rangle = 0\} = 0$. Thus by (1), $\mathrm{Lk}(\Gamma, n) = 2$.

It remains to find ρ. Let $t_1 = 7\pi/6$, $t_2 = 11\pi/6$, $t_3 = \pi/6$, $t_4 = 5\pi/6$, and $t_5 = \pi/2$. Then the origin o of $\mathbf{R}^3$ is contained in the interior of the convex hull of $p_i := T(t_i)$, see Figure 3. One may also note that the pairs (p_1, p_2) and (p_3, p_4) are

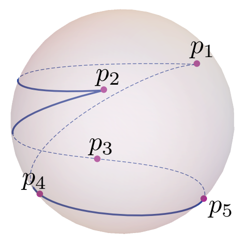

Figure 3.

symmetric with respect to the yz-pane, while p_5 lies on the yz-plane. Furthermore, p_1 and p_2 lie on the cusps of T which correspond to the inflections of n. Set $p_i' := \int \phi_i T$ for positive functions $\phi_i \colon \mathbf{S}^1 \to \mathbf{R}$ with $\int \phi_i = 1$. If ϕ_i are concentrated near t_i then $|p_i' - p_i|$ are small. So o lies in the interior of the convex hull of p_i'. For instance we may set $\phi_i(t) := (1.1+\cos(t-t_i))^{10} / \int (1.1+\cos(t-t_i))^{10}$. Then there exist constants $c_i > 0$ such that $\sum c_i p_i' = o$. In particular we may set $c_1 = c_2 = 1$, $c_3 = c_4 \approx 0.0815$, and $c_5 \approx 0.7465$. Then we set $\rho := \sum c_i \phi_i$.

5. Further Observations

Here we include some observations on the structure of Γ and its projections Γ_u in the case where n is injective.

Note 5.1. When n is injective, Γ must have inflections. To see this note that the geodesic curvature of n in $\mathbf{S}^2$ is given by

$$\widetilde{\kappa}_g := \frac{\langle n'', n \times n'\rangle}{|n'|^3} = \frac{\langle -\tau_g' n^\perp - \tau_g (n^\perp)', \tau_g T\rangle}{\tau_g^3} = \frac{\langle -(n^\perp)', T\rangle}{\tau_g} = \frac{\kappa_g}{\tau_g}, \tag{9}$$

and recall that $|\kappa_g| = \kappa$. Thus if $\kappa \neq 0$, then κ_g has a fixed sign. Consequently n bounds a geodesically convex domain Ω in $\mathbf{S}^2$. In particular n lies in a hemisphere centered at a point of Ω. It follows that T lies in the opposite hemisphere, since T is traced by the centers of oriented great circles tangent to n. But since Γ is closed, the origin of $\mathbf{R}^3$ must lie in the relative interior of the convex hull of T. Hence T must be a great circle, which in turn implies that Γ is a planar curve, or n is constant, which is a contradiction.

Note 5.2. An alternative argument for showing that injectivity of n forces inflections along Γ has been given by Kovaleva [27]. Since that work has been published only in Russian, we include the argument here. Let $\widetilde{\Gamma} := n(\Gamma)$, and A_1, A_2 be the areas of the components of $\mathbf{S}^2 \setminus \widetilde{\Gamma}$. Then the Gauss-Bonnet theorem yields that

$$\left| \int_{\widetilde{\Gamma}} \widetilde{\kappa}_g \right| = \frac{1}{2}|A_1 - A_2| < 2\pi. \tag{10}$$

Furthermore, since $|n'| = \tau_g$ it follows from (9) that

$$\int_{\widetilde{\Gamma}} \widetilde{\kappa}_g = \int_0^{2\pi} \widetilde{\kappa}_g(t)|n'(t)|dt = \text{sign}(\tau_g) \int_0^{2\pi} \frac{\kappa_g(t)}{\tau_g(t)} \tau_g(t) dt = \text{sign}(\tau_g) \int_\Gamma \kappa_g. \tag{11}$$

So $|\int \kappa_g| < 2\pi$. Now suppose that κ_g does not change sign. Then $\int \kappa = \int |\kappa_g| = |\int \kappa_g| < 2\pi$. But since Γ is a closed curve in $\mathbf{R}^3$, $\int \kappa \geq 2\pi$ by Fenchel's theorem. Hence we arrive at a contradiction.

Note 5.3. When n is injective,

$$\left(\int_\Gamma \kappa_g \right)^2 + \left(\int_\Gamma \tau_g \right)^2 > 4\pi^2. \tag{12}$$

To see this let L denote the length of $\widetilde{\Gamma} = n(\Gamma)$, and A be the area of the region bounded by $\widetilde{\Gamma}$ into which $n \times n' = -\tau_g n \times n^\perp = \tau_g T$ points. Assuming $\tau_g > 0$, we have $L = \int_\Gamma |n'| = \int_\Gamma \tau_g$. Furthermore, by Gauss-Bonnet theorem and (11) $A = 2\pi - \int_{\widetilde{\Gamma}} \widetilde{\kappa}_g = 2\pi - \int_\Gamma \kappa_g$. By the isoperimetric inequality on $\mathbf{S}^2$, $L^2 \geq 4\pi A - A^2$ with equality only if $\widetilde{\Gamma}$ is a circle. Since $\widetilde{\Gamma}$ has inflections, as discussed in Note 5.1, $\widetilde{\Gamma}$ cannot be a circle. Thus $L^2 > 4\pi A - A^2$, which yields (12).

Note 5.4. If n is injective and $\langle u, n \rangle \neq 0$, or M is locally a graph over a plane orthogonal to u, then the absolute rotation index of Γ_u is 1 (which was precisely the case in Example 4.2). To see this suppose again that $u = (0, 0, 1)$ and consider the rescaling $(x, y, z) \mapsto (x, y, \lambda z)$ as in the proofs of Theorems 1.3 and 1.4. Then, as $\lambda \to 0$, $\int \kappa_g$ converges to the total geodesic curvature of Γ_u. But $\int \kappa_g = \pm \int \widetilde{\kappa}_g$ by (11). Furthermore, $\int \widetilde{\kappa}_g$ converges to $\pm 2\pi$ by (10), since $\widetilde{\Gamma}$ converges to u or $-u$. So the total geodesic curvature of Γ_u is $\pm 2\pi$, which means the rotation index is ± 1.

Note 5.5. In [27] Kovaleva studied Problem 1.1 in the case where Γ has only finitely many inflections, and claimed that when n is injective, $\text{Lk}(\Gamma, n)$ is equal to half the number of inflections. But as we pointed out in Note 5.1, inflections of Γ correspond to inflections of n in $\mathbf{S}^2$. As Figure 2(b) shows, n has only two inflections in Example 4.2, whereas $\text{Lk}(\Gamma, n) = 2$ as we discussed above. Thus Example 4.2 contradicts Kovaleva's claim.

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