

A CONVERSE TO THE ARCHIMEDES SPHERE THEOREM

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ABSTRACT. We show that a convex surface in Euclidean 3-space is a sphere if the area cut off from it by a slab of some fixed width depends only on the direction of the slab. This result generalizes theorems of Blaschke and Stamm, which assumed the slab-area property for every width. It holds also for any smooth closed embedded surface. The proof is based on an averaging identity, which relates slab areas to the Newtonian potential of the surface, the electrostatic characterization of spheres by Mendez and Reichel, and analytic microlocal properties of the Radon transform, which are used to show that the slab-area property forces real analyticity, via theorems of Kashiwara and Boman.

1. INTRODUCTION

Archimedes showed that the area cut off from a sphere by a slab depends only on the width of the slab [1]. Blaschke [2] proved a converse: a smooth strictly convex surface whose slab areas depend only on the width of the slab is a sphere. Stamm [26] extended this result by allowing the areas to depend also on the direction of the slabs. A problem posed in [10] asks whether it suffices to consider a single width. We show that the answer is yes.

A *convex surface* Γ is the boundary of a *convex body*, i.e., a compact convex set $K \subset \mathbf{R}^3$ with nonempty interior. A *slab* is the closed region between two parallel planes, and its width is the distance between them. For a compact surface Γ , we say that Γ has the *slab-area property* for $h > 0$ if, for each direction $u \in \mathbf{S}^2$, all slabs of width h orthogonal to u whose boundary planes both intersect Γ cut off the same area from Γ . Let $\text{width}(\Gamma)$ denote the minimum distance between two parallel supporting planes of Γ .

Theorem 1.1. *If Γ is a convex surface with the slab-area property for some $0 < h < \text{width}(\Gamma)$, then Γ is a sphere.*

The convexity assumption may be omitted when there is sufficient regularity:

Theorem 1.2. *Let $\Gamma \subset \mathbf{R}^3$ be a $\mathcal{C}^3$ closed embedded surface. If Γ has the slab-area property for some $0 < h < \text{width}(\Gamma)$, then Γ is a sphere.*

The proof of Theorem 1.1 begins with an averaging identity (Section 2): if $x \in K$ lies at distance $> h/2$ from Γ , the average area of the width- h slabs centered at

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x is $2\pi h$ times the Newtonian potential $U(x)$ of Γ . When h is less than twice the inradius of Γ , this forces U to be constant on Γ , and the electrostatic characterization of spheres by Mendez and Reichel [19] quickly establishes Theorem 1.1.

For general h , we first treat the real analytic case (Section 3) by considering the area density of Γ with respect to height in each direction. The slab-area property makes these densities periodic. Real analyticity allows them to be continued to a bounded periodic density. Averaging this continuation over directions produces a bounded real analytic function on $\mathbf{R}^3$ which agrees with U on the interior of K . Since U is harmonic there, analyticity implies that this function is harmonic on $\mathbf{R}^3$, and hence constant by Liouville's theorem.

To complete the proof, it remains to show that the slab-area property forces real analyticity. We first prove, by elementary means, that it forces $\mathcal{C}^1$ regularity (Section 4). Next we show that a $\mathcal{C}^1$ surface satisfying the condition is real analytic (Section 5), using analytic microlocal properties of the Radon transform, Kashiwara's watermelon theorem [15, 25], and a recent regularity result of Boman [3].

To prove Theorem 1.2, we show that the periodicity of the area densities forces the mean curvature of Γ to be nonpositive inside its convex hull, which by the maximum principle makes Γ convex (Section 6). Theorem 1.1 then applies. See [6, 9, 14, 16, 17, 21, 22, 28] for other studies related to the Archimedean property of the sphere. Theorems 1.1 and 1.2 are the first known results for a single width.

Note 1.3. If the common area in the slab-area property is the same in all directions, as in the original question [10], then Theorems 1.1 and 1.2 hold under the weaker assumption $h < \text{diam}(\Gamma)$. Indeed, if $\text{width}(\Gamma) \leq h < \text{diam}(\Gamma)$, continuity yields a direction in which the width is h , so the common area is the total area of Γ . But in any direction of width greater than h , a slab of width h omits a cap of Γ , a contradiction.

2. NEWTONIAN POTENTIAL

Throughout Sections 2–5, $K \subset \mathbf{R}^3$ is a convex body and $\Gamma := \partial K$, unless stated otherwise. For $x \in K$, let

$$U(x) := \frac{1}{4\pi} \int_{y \in \Gamma} \frac{1}{|x - y|}$$

be the *Newtonian potential* of Γ at x . The proof of Theorem 1.1 is based on the following characterization of the sphere.

Lemma 2.1 (Mendez–Reichel [19]). *If U is constant on Γ , then Γ is a sphere.*

This result had been conjectured by Gruber [12, p. 353] and proved by Reichel [20] for smooth surfaces. We also need the following fact which is proved by a standard argument, cf. [18, p. 59]. Let $\text{int}(K)$ denote the interior of K .

Lemma 2.2. *U is continuous on K and harmonic on $\text{int}(K)$.*

Proof. Harmonicity follows by differentiating under the integral sign. To prove continuity, let $B_\rho(x)$ be the ball of radius ρ . By monotonicity of surface area for nested

convex bodies, $f(t) := |\Gamma \cap B_t(x)| \leq |\partial B_t(x)| = 4\pi t^2$. The Lebesgue–Stieltjes measure df is the push-forward of area measure on Γ by the map $y \mapsto |x - y|$. Hence

$$\int_{y \in \Gamma \cap B_\rho(x)} \frac{1}{|x - y|} = \int_0^\rho \frac{df(t)}{t}.$$

Since $f(t) = O(t^2)$ as $t \rightarrow 0^+$, integration by parts gives

$$\int_0^\rho \frac{df(t)}{t} = \frac{f(\rho)}{\rho} + \int_0^\rho \frac{f(t)}{t^2} dt \leq 8\pi\rho.$$

Now let U_ρ be defined as U , but with $|x - y|^{-1}$ replaced by $\min\{\rho^{-1}, |x - y|^{-1}\}$. Then U_ρ is continuous on K , since its integrand is bounded and continuous, and $0 \leq U - U_\rho \leq 2\rho$ on K by the last estimate. Thus U is continuous, being a uniform limit of continuous functions. $\square$

For $h > 0$, $x \in \mathbf{R}^3$, and $u \in \mathbf{S}^2$, let $S_h(x, u) := \{y \in \mathbf{R}^3 : |\langle y - x, u \rangle| \leq h/2\}$ be the slab of width h with midplane through x and normal u . For $h > 0$, set $K_h := \{x \in K : \text{dist}(x, \Gamma) > h/2\}$. We write $|E|$ for the area of a set $E \subset \Gamma$.

Lemma 2.3 (The averaging identity). *For every $x \in K_h$,*

$$\frac{1}{4\pi} \int_{u \in \mathbf{S}^2} |\Gamma \cap S_h(x, u)| = 2\pi h U(x).$$

Proof. Fix $x \in K_h$. For $y \in \Gamma$, let $Z(y) := \{u \in \mathbf{S}^2 : |\langle y - x, u \rangle| \leq h/2\}$ be the zone of $\mathbf{S}^2$ between the two planes orthogonal to $y - x$ at distance $h/(2|x - y|)$ from the origin. Since $|x - y| > h/2$, this distance is less than 1, so both planes meet $\mathbf{S}^2$, and the zone has width $h/|x - y|$. By the theorem of Archimedes,

$$|Z(y)| = \frac{2\pi h}{|x - y|}.$$

Next note that $y \in S_h(x, u)$ if and only if $u \in Z(y)$, since both conditions state that $|\langle y - x, u \rangle| \leq h/2$. Hence, for each $u \in \mathbf{S}^2$,

$$|\Gamma \cap S_h(x, u)| = \int_{y \in \Gamma} \mathbf{1}_{S_h(x, u)}(y) = \int_{y \in \Gamma} \mathbf{1}_{Z(y)}(u).$$

Integrating over u and applying Fubini's theorem, we obtain

$$\begin{aligned} \int_{u \in \mathbf{S}^2} |\Gamma \cap S_h(x, u)| &= \int_{u \in \mathbf{S}^2} \int_{y \in \Gamma} \mathbf{1}_{Z(y)}(u) \\ &= \int_{y \in \Gamma} \int_{u \in \mathbf{S}^2} \mathbf{1}_{Z(y)}(u) = \int_{y \in \Gamma} |Z(y)| = 8\pi^2 h U(x), \end{aligned}$$

which completes the proof. $\square$

The above results quickly yield Theorem 1.1 for widths which are not too large. From now on, through Section 5, we assume that Γ has the slab-area property for some fixed $0 < h < \text{width}(\Gamma)$. The *inradius* $\text{inrad}(\Gamma)$ of Γ is the radius of a largest ball contained in K .

Proposition 2.4. *If $h < 2 \text{inrad}(\Gamma)$, then Γ is a sphere.*

Proof. Since K_h is nonempty, for each fixed $u \in \mathbf{S}^2$ the quantity $|\Gamma \cap S_h(x, u)|$ is independent of $x \in K_h$. Hence its average over u is independent of x . By Lemma 2.3, U is constant on K_h . Since U is harmonic on the connected set $\text{int}(K)$ and is constant on the nonempty open set K_h , unique continuation shows that U is constant on $\text{int}(K)$. By Lemma 2.2, it has the same constant value on Γ , and Lemma 2.1 completes the proof. $\square$

3. THE REAL ANALYTIC CASE

Here we show that Theorem 1.1 holds when Γ is real analytic. We begin by recording some facts which hold for any convex surface. For $u \in \mathbf{S}^2$, let $\phi_u(y) := \langle y, u \rangle$ be the *height function* in the direction u , on $\mathbf{R}^3$ and, by restriction, on Γ . The *support function* of Γ is given by $p(u) := \max_{\Gamma} \phi_u$. For $-p(-u) < t < p(u)$, let

$$\Gamma_{u,t} := \{y \in \Gamma : \phi_u(y) = t\}$$

be the level sets of ϕ_u on Γ . Since Γ is locally Lipschitz, it has a unique outward unit normal ν almost everywhere, and there the tangential gradient of ϕ_u is $\nabla \phi_u(y) = u - \langle \nu(y), u \rangle \nu(y)$. Thus

$$(1) \quad R(u, t) := \int_{y \in \Gamma_{u,t}} \frac{1}{|\nabla \phi_u(y)|} = \int_{y \in \Gamma_{u,t}} \frac{1}{\sqrt{1 - \langle \nu(y), u \rangle^2}}$$

is defined for almost every t , and the coarea formula gives

$$(2) \quad |\{y \in \Gamma : a \leq \phi_u(y) \leq b\}| = \int_a^b R(u, t) dt$$

for $-p(-u) < a < b < p(u)$. We call $R(u, t)$ the *area density* in direction u and set $R(u, t) = 0$ for $t < -p(-u)$ or $t > p(u)$. Let $D := \{(u, t) \in \mathbf{S}^2 \times \mathbf{R} : -p(-u) < t < p(u)\}$.

Lemma 3.1. *If Γ is $\mathcal{C}^1$, R is continuous on D . If Γ is real analytic, R is real analytic on D .*

Proof. Fix $(u_0, t_0) \in D$, and choose a point $c_0 \in \text{int}(K)$ with $\phi_{u_0}(c_0) = t_0$. For (u, t) near (u_0, t_0) , the plane $\{\phi_u = t\}$ meets the fixed line $c_0 + \mathbf{R}u_0$ at

$$c(u, t) := c_0 + \frac{t - \langle c_0, u \rangle}{\langle u_0, u \rangle} u_0.$$

Thus $c(u, t)$ depends real analytically on (u, t) and, after shrinking the neighborhood, belongs to $\text{int}(K)$. Choose a real analytic orthonormal frame $e_1(u), e_2(u)$ of $u^\perp$ defined on a neighborhood of u_0 in $\mathbf{S}^2$, and put $e(u, \theta) := \cos \theta e_1(u) + \sin \theta e_2(u)$. For every nearby (u, t) and θ , the ray $c(u, t) + re(u, \theta)$ meets Γ in a unique point

$$y(u, t, \theta) = c(u, t) + r(u, t, \theta)e(u, \theta).$$

The intersection is transverse, since $c(u, t) \in \text{int}(K)$ and a supporting plane at $y(u, t, \theta)$ separates $c(u, t)$ strictly from that point. The implicit function theorem shows that r is $\mathcal{C}^1$ locally, and real analytic if Γ is real analytic. Uniqueness makes the local solutions agree on overlaps, and compactness of the parameter circle gives

a common neighborhood of (u_0, t_0) . Thus $y(u, t, \theta)$ is a $\mathcal{C}^1$ parametrization of $\Gamma_{u,t}$ depending $\mathcal{C}^1$ on (u, t) . Since $\partial_\theta y \neq 0$,

$$R(u, t) = \int_0^{2\pi} \frac{|\partial_\theta y(u, t, \theta)|}{\sqrt{1 - \langle \nu(y(u, t, \theta)), u \rangle^2}} d\theta$$

is continuous. If Γ is real analytic, then $\partial_\theta y$ is real analytic and nonzero, so $|\partial_\theta y| = \sqrt{\langle \partial_\theta y, \partial_\theta y \rangle}$ is real analytic. The denominator is likewise real analytic. Hence the integrand is real analytic. Compactness of the parameter circle supplies uniform local analytic estimates in (u, t) , so integration also gives real analyticity of R . $\square$

Since $h < \text{width}(\Gamma) \leq p(u) + p(-u)$, for $-p(-u) < t < p(u) - h$ the planes at heights t and $t+h$ both meet Γ , so by the slab-area property and (2) $\int_t^{t+h} R(u, s) ds$ is independent of t . Hence, by Lebesgue's differentiation theorem,

$$(3) \quad R(u, t+h) = R(u, t)$$

for almost every $-p(-u) < t < p(u) - h$, and for every such t when R is continuous.

Proposition 3.2. *If Γ is real analytic, then it is a sphere.*

Proof. We extend R periodically. For $(u, t) \in \mathbf{S}^2 \times \mathbf{R}$, choose $k \in \mathbf{Z}$ so that $-p(-u) < t + kh < p(u)$, which is possible since $p(u) + p(-u) > h$, and set

$$\tilde{R}(u, t) := R(u, t + kh).$$

This is well defined: if two choices of k are possible, repeated use of (3) gives the same value. Since p is continuous, the same k works in a neighborhood of any fixed (u, t) ; Lemma 3.1 therefore shows that $\tilde{R}$ is real analytic. It is h -periodic in t , so it descends to a real analytic function on the compact manifold $\mathbf{S}^2 \times (\mathbf{R}/h\mathbf{Z})$ and in particular is bounded. Define, for $x \in \mathbf{R}^3$,

$$V(x) := \frac{1}{8\pi^2} \int_{u \in \mathbf{S}^2} \tilde{R}(u, \phi_u(x)).$$

The function V is bounded and real analytic on $\mathbf{R}^3$. Indeed, compactness of $\mathbf{S}^2 \times (\mathbf{R}/h\mathbf{Z})$ and real analyticity give constants $C, L > 0$ such that $|\partial_t^m \tilde{R}(u, t)| \leq CL^m m!$ for all $m \geq 0$, and differentiation under the integral gives the corresponding analytic estimates for V .

We claim that $V = U$ on $\text{int}(K)$. Fix $x \in \text{int}(K)$ and put $d := \text{dist}(x, \Gamma)$. If $0 < \varepsilon < 2d$, then $x \in K_\varepsilon$, so Lemma 2.3, with h replaced by ε , and (2) give

$$2\pi U(x) = \frac{1}{4\pi} \int_{u \in \mathbf{S}^2} \frac{1}{\varepsilon} \int_{\phi_u(x) - \varepsilon/2}^{\phi_u(x) + \varepsilon/2} R(u, t) dt.$$

As $\varepsilon \rightarrow 0^+$, the inner averages converge uniformly in u to $R(u, \phi_u(x))$, since these parameters range over a compact subset of D . Furthermore, since $(u, \phi_u(x)) \in D$ for every $u \in \mathbf{S}^2$, we have $\tilde{R}(u, \phi_u(x)) = R(u, \phi_u(x))$. Consequently,

$$2\pi U(x) = \frac{1}{4\pi} \int_{u \in \mathbf{S}^2} R(u, \phi_u(x)) = \frac{1}{4\pi} \int_{u \in \mathbf{S}^2} \tilde{R}(u, \phi_u(x)) = 2\pi V(x),$$

as desired. Now since U is harmonic on $\text{int}(K)$, so is V . But V is real analytic on $\mathbf{R}^3$, hence ΔV is real analytic and vanishes on the nonempty open set $\text{int}(K)$. Therefore $\Delta V = 0$ on $\mathbf{R}^3$. Thus V is a bounded harmonic function on $\mathbf{R}^3$, and Liouville's theorem yields that V is constant. Hence U is constant on $\text{int}(K)$ and, by Lemma 2.2, on Γ . Lemma 2.1 completes the proof. $\square$

4. BASIC REGULARITY

Here we show that the slab-area property forces Γ to be $\mathcal{C}^1$. The proof is based on a uniform estimate for the areas of *caps* of Γ , i.e., portions cut off by a plane. First we record the following basic fact.

Lemma 4.1. *For any supporting plane P of Γ , $|P \cap \Gamma| = 0$.*

Proof. Let u be the outward unit normal of P , so that P is at height $p(u)$. For small $\varepsilon > 0$, the slab between the heights $p(u) - h$ and $p(u)$ and its translate between $p(u) - h - \varepsilon$ and $p(u) - \varepsilon$ cut off the same area from Γ . Removing their common part, we obtain

$$|\Gamma \cap \{p(u) - \varepsilon < \phi_u \leq p(u)\}| = |\Gamma \cap \{p(u) - h - \varepsilon \leq \phi_u < p(u) - h\}|.$$

As $\varepsilon \rightarrow 0^+$, the sets on the right decrease to the empty set, while those on the left decrease to $P \cap \Gamma$; hence $|P \cap \Gamma| = 0$. $\square$

For $-p(-u) < t \leq p(u)$, let $A(u, t) := |\{y \in \Gamma : \phi_u(y) \geq t\}|$ be the area of the cap of Γ cut off by the plane orthogonal to u at height t , on the side of u . Thus $A(u, p(u) - s)$ is the area of the cap of depth s below the supporting plane with outward normal u . By Lemma 4.1, (2) also holds for $b = p(u)$; in particular

$$(4) \quad A(u, t) = \int_t^{p(u)} R(u, s) ds.$$

Let $\text{width}_u(\Gamma) := p(u) + p(-u)$ denote the width of Γ in the direction u .

Lemma 4.2. *There are constants $m, M > 0$ such that $ms \leq A(u, p(u) - s) \leq Ms$ for every $u \in \mathbf{S}^2$ and $0 < s < \text{width}_u(\Gamma)$.*

Proof. Put $d := \text{diam}(\Gamma)$, $r := \text{inrad}(\Gamma)$, and $\delta := (\text{width}(\Gamma) - h)/2 > 0$. Let $c(u) := (p(u) - p(-u))/2$ and $I_u := [c(u) - h/2, c(u) + h/2]$, so that $-p(-u) + \delta \leq t \leq p(u) - \delta$ for $t \in I_u$. At a point $y \in \Gamma_{u,t}$ where there is a unique outward normal ν , we claim

$$(5) \quad |\nabla \phi_u(y)| \geq \delta/d.$$

If $\langle \nu(y), u \rangle \geq 0$, choose $z \in K$ with $\phi_u(z) = p(u)$. Then $\langle z - y, \nu(y) \rangle \leq 0$, since K lies below the supporting plane of Γ at y , and so $\delta \leq p(u) - t = \langle z - y, u \rangle \leq \langle z - y, \nabla \phi_u(y) \rangle \leq d |\nabla \phi_u(y)|$, as desired. When $\langle \nu(y), u \rangle < 0$, apply the same argument with u replaced by $-u$. Thus (5) holds.

We may assume that $B_r(0) \subset K$, after a translation. Then $p(u) \leq d$, and the planar section $K \cap \{\phi_u = t\}$ has diameter at most d . It also contains a disk of radius at least $r\delta/d$. This follows by taking the convex hull of $B_r(0)$ with a point $z \in K$ satisfying $\phi_u(z) = p(u)$, or with the corresponding point in direction $-u$ when

$t < 0$. Consequently, if $L(\Gamma_{u,t})$ denotes the length of $\Gamma_{u,t}$, which is the boundary of the planar section, then $2\pi r\delta/d \leq L(\Gamma_{u,t}) \leq 2\pi d$. On the other hand, by (5), the integrand in (1) lies between 1 and d/δ on $\Gamma_{u,t}$. Hence

$$m := 2\pi r\delta/d \leq L(\Gamma_{u,t}) \leq R(u,t) \leq L(\Gamma_{u,t})d/\delta \leq 2\pi d^2/\delta =: M$$

for almost every $t \in I_u$. Since I_u has length h , every $t \in (-p(-u), p(u))$ differs from a point of I_u by an integer multiple of h . So by (3) the same bounds hold for almost every such t . Hence (4) yields the desired estimate. $\square$

For the next two arguments, it is convenient to fix $q \in \Gamma$ and, for $u \in \mathbf{R}^3$ and $z \in K$, define the function

$$f_u(z) := \langle q - z, u \rangle.$$

Then f_u depends linearly on u , and a nonzero vector u is a supporting normal of Γ at q if and only if $f_u \geq 0$ on K .

Lemma 4.3. Γ is strictly convex.

Proof. Suppose that Γ contains a segment $q + [-\ell/2, \ell/2]e$, where $|e| = 1$ and $\ell > 0$. Every supporting normal at q is perpendicular to e . We show that these normals form an open set in the circle $S := \mathbf{S}^2 \cap e^\perp$. Fix a supporting unit normal n at q , so that $f_n \geq 0$ on K , and let $v := n \times e$. We claim that, for every $z \in K$,

$$(6) \quad |f_v(z)| \leq (2M/\ell)f_n(z).$$

Suppose first that $f_n(z) > 0$, and choose $0 < s < f_n(z)$. The cap body $K \cap \{0 \leq f_n(y) \leq s\}$ contains the triangle with vertices $q - \ell e/2$, $q + \ell e/2$, and $q + (s/f_n(z))(z - q)$. Its projection onto $n^\perp$ has area $\ell s |f_v(z)| / (2f_n(z))$. Every line parallel to n which meets the cap body $K \cap \{0 \leq f_n \leq s\}$ also meets the surface cap $\Gamma \cap \{0 \leq f_n \leq s\}$. Thus the orthogonal projection of the surface cap onto $n^\perp$ contains the projection of the triangle above. Since orthogonal projection does not increase area, Lemma 4.2 yields

$$Ms \geq A(n, p(n) - s) \geq \ell s |f_v(z)| / (2f_n(z)),$$

as desired. If $f_n(z) = 0$, then z lies in the supporting face of K with normal n . If $f_v(z) \neq 0$, that face would contain, together with the given segment, a triangle of positive area, contrary to Lemma 4.1. Thus $f_v(z) = 0$, and (6) holds. Now, if $|\varepsilon| < \ell/(2M)$, then for every $z \in K$,

$$f_{n+\varepsilon v}(z) = f_n(z) + \varepsilon f_v(z) \geq (1 - 2M|\varepsilon|/\ell) f_n(z) \geq 0.$$

Hence $(n + \varepsilon v)/|n + \varepsilon v|$ is a supporting normal. So the set of supporting unit normals at q is open in S . It is also nonempty and closed, and therefore equals S . This forces K to lie in the line $q + \mathbf{R}e$, contrary to K having nonempty interior. $\square$

Proposition 4.4. Γ is $\mathcal{C}^1$.

Proof. It suffices to show that Γ has a unique supporting plane at each point [23, Thm. 2.2.4]. Suppose to the contrary that $q \in \Gamma$ has two distinct outward unit normals n_1, n_2 . Set $n := (n_1 + n_2)/|n_1 + n_2|$ and $v := (n_1 - n_2)/|n_1 - n_2|$. Then $n \perp v$, and $n_1 = c_1 n + c_2 v$, $n_2 = c_1 n - c_2 v$ for some $c_1, c_2 > 0$. Since $f_{n_1} = c_1 f_n + c_2 f_v$

and $f_{n_2} = c_1 f_n - c_2 f_v$ are nonnegative on K , we have $|f_v| \leq \alpha f_n$ on K , where $\alpha := c_1/c_2$. In particular $f_n \geq 0$ on K , so n is a supporting normal at q . Consider the cap $C_s := K \cap \{f_n(z) \leq s\}$, and let $e := n \times v$. By Lemma 4.3, the supporting face with normal n consists only of q . Hence there is a function $\varepsilon(s) \rightarrow 0$ as $s \rightarrow 0^+$ such that $|f_e| \leq \varepsilon(s)$ on C_s . Thus C_s lies in a rectangular box with side lengths s , $2\alpha s$, and $2\varepsilon(s)$. By monotonicity of surface area for nested convex bodies, $A(n, p(n) - s) \leq |\partial C_s| \leq 4\alpha s^2 + (8\alpha + 4)s\varepsilon(s)$. It follows that $A(n, p(n) - s)/s \rightarrow 0$, contradicting the lower bound in Lemma 4.2. $\square$

5. PROOF OF THEOREM 1.1

In view of Propositions 3.2 and 4.4, to prove Theorem 1.1 it remains to show:

Proposition 5.1. *If Γ is $\mathcal{C}^1$, then it is real analytic.*

To prove the proposition we use the *analytic wave front set* $\text{WF}_A(\mu)$ of a distribution μ , which records the points and directions in which μ fails to be real analytic; see [13, Chap. 8 & 9], [24]. Throughout this section μ denotes the area measure of Γ , regarded as a compactly supported distribution on $\mathbf{R}^3$, $\langle \mu, \psi \rangle := \int_{y \in \Gamma} \psi(y)$, whose support is Γ . We call a nonzero vector ξ a *singular direction* of μ at q if $(q, \xi) \in \text{WF}_A(\mu)$. We need the following facts:

Lemma 5.2 ([13, Thm. 8.4.5]). *Singular directions of μ occur only at points of Γ .*

Lemma 5.3 (Kashiwara's Watermelon Theorem [13, Cor. 9.6.8]). *Let $q \in \Gamma$. If $\xi \notin \mathbf{R}\nu(q)$ is a singular direction of μ at q , then so is every vector in the open half-plane of $\text{span}\{\nu(q), \xi\}$ bounded by $\mathbf{R}\nu(q)$ which contains ξ .*

Lemma 5.4 (Boman [3, Thm. 1]). *Let $q \in \Gamma$. If no singular direction of μ at q is tangent to Γ , then Γ is real analytic near q .*

The slab-area property is a condition on the area densities $R(u, \cdot)$; the Radon transform is what ties them to the singularities of μ . For a smooth compactly supported function ψ on $\mathbf{R}^3$, its *Radon transform* is

$$\mathcal{R}\psi(u, t) := \int_{\phi_u(y)=t} \psi(y),$$

the integral of ψ over the plane orthogonal to u at height t , for $(u, t) \in \mathbf{S}^2 \times \mathbf{R}$. It extends to compactly supported distributions by duality, and for μ the coarea formula gives, for $(u, t) \in D$,

$$\mathcal{R}\mu(u, t) = R(u, t).$$

The wave front set of $\mathcal{R}\mu$ lies in the cotangent bundle $T^*(\mathbf{S}^2 \times \mathbf{R})$. There is a natural identification of $T^*_{(u,t)}(\mathbf{S}^2 \times \mathbf{R})$ with $u^\perp \times \mathbf{R}$, and we write its elements as $(u, t; \eta, \tau)$. Let $P_u q := q - \phi_u(q)u$ be the orthogonal projection of q onto $u^\perp$. The wave front correspondence for the Radon transform [5, Thm. 1.2 & Lem. 2.5], which goes back to Boman–Quinto [4], gives

$$(7) \quad (q, u) \in \text{WF}_A(\mu) \iff (u, \phi_u(q); -P_u q, 1) \in \text{WF}_A(\mathcal{R}\mu).$$

Thus a singular direction u of μ at q corresponds to a singularity of $\mathcal{R}\mu$ at the plane through q orthogonal to u .

Lemma 5.5. *Let $q \in \Gamma$, $u \in \mathbf{S}^2$ be a singular direction of μ at q with $u \neq \pm\nu(q)$, and q' be one of the points $q \pm hu$. If the plane orthogonal to u through q' meets $\text{int}(K)$, then $q' \in \Gamma$, and u is a singular direction of μ at q' .*

Proof. Since $u \neq \pm\nu(q)$, the plane orthogonal to u through q is not a supporting plane and hence meets $\text{int}(K)$. The plane through q' meets $\text{int}(K)$ by hypothesis. Hence $\phi_u(q)$ and $\phi_u(q')$ both lie in $(-p(-u), p(u))$. Let t be the larger of the two, so that $-p(-u) + h < t < p(u)$. By (3), $\mathcal{R}\mu(u, s) = \mathcal{R}\mu(u, s - h)$ whenever $-p(-u) + h < s < p(u)$. Since $\mathcal{R}\mu$ coincides with its translate $\mathcal{R}\mu(\cdot, \cdot - h)$ on the open set $\{(u, s) : -p(-u) + h < s < p(u)\}$, and wave front sets are invariant under translation,

$$(u, t; \eta, \tau) \in \text{WF}_A(\mathcal{R}\mu) \iff (u, t - h; \eta, \tau) \in \text{WF}_A(\mathcal{R}\mu).$$

Since $P_u q' = P_u q$ and $\{t, t - h\} = \{\phi_u(q), \phi_u(q')\}$, (7) shows that u is a singular direction of μ at q if and only if it is one at q' . Thus u is a singular direction at q' , and so $q' \in \Gamma$ by Lemma 5.2. $\square$

Proof of Proposition 5.1. Fix $q \in \Gamma$ and put $n := \nu(q)$. By Lemma 5.4, it suffices to show that no singular direction of μ at q is tangent to Γ . Suppose, to the contrary, that $e \in \mathbf{S}^2$ is such a direction. Put $w := n \times e$, and

$$u_\theta := \cos \theta n + \sin \theta e, \quad u'_\theta := -\sin \theta n + \cos \theta e,$$

so that (u_θ, u'_θ, w) is an orthonormal frame for every $\theta \in \mathbf{R}$. We proceed in four steps. First, we apply Lemmas 5.3 and 5.5 at q to show that Γ contains the curve $q_\theta := q - hu_\theta$ for small $\theta > 0$, with u_θ singular at q_θ . Second, we apply Lemma 5.3 at q_θ to obtain singular directions in $\text{span}\{u_\theta, w\}$ close to u_θ . Third, we apply Lemma 5.5 uniformly in θ to the singular directions obtained in the second step, producing a two-parameter family of points of Γ which contains q when the second parameter vanishes. Finally, fixing that parameter at a small nonzero value and differentiating with respect to θ yields tangent vectors of Γ near q which are not orthogonal to n , contradicting the continuity of ν .

Step 1. We first show that $q_\theta \in \Gamma$ for small $\theta > 0$. For $0 < \theta < \pi$, u_θ lies in the open half-plane of $\text{span}\{n, e\}$ bounded by $\mathbf{R}n$ which contains e , so u_θ is a singular direction of μ at q by Lemma 5.3. The plane orthogonal to u_θ through q_θ meets $\text{int}(K)$ if and only if $-p(-u_\theta) < \langle q, u_\theta \rangle - h$, since $\langle q, u_\theta \rangle - h < p(u_\theta)$ holds trivially. For $\theta = 0$ this inequality reads $-p(-n) < p(n) - h$, which holds because $h < \text{width}(\Gamma) \leq \text{width}_n(\Gamma) = p(n) + p(-n)$. Hence, by continuity, there is $\theta_1 \in (0, \pi)$ such that the inequality holds for $0 \leq \theta < \theta_1$. Lemma 5.5 now shows that $q_\theta \in \Gamma$, and that u_θ is a singular direction of μ at q_θ , for $0 < \theta < \theta_1$.

Step 2. We next perturb u_θ through singular directions of μ at q_θ which leave the plane $\text{span}\{n, e\}$. Put $\nu_\theta := \nu(q_\theta)$. The curve $\theta \mapsto q_\theta$ lies in Γ and has velocity $-hu'_\theta$, so $\langle \nu_\theta, u'_\theta \rangle = 0$, that is, $\nu_\theta \in \text{span}\{u_\theta, w\}$. Moreover $\nu_\theta \neq \pm u_\theta$, because the plane orthogonal to u_θ through q_θ meets $\text{int}(K)$ and so is not the tangent plane of Γ at q_θ . Thus u_θ is a singular direction of μ at q_θ not parallel to ν_θ , and

$\text{span}\{\nu_\theta, u_\theta\} = \text{span}\{u_\theta, w\}$. Let $\alpha(\theta) > 0$ be the angle between u_θ and the line $\mathbf{R}\nu_\theta$, and for $\varphi \in \mathbf{R}$ put $v_{\theta,\varphi} := \cos \varphi u_\theta + \sin \varphi w$. For $|\varphi| < \alpha(\theta)$, $v_{\theta,\varphi}$ lies in the open half-plane of $\text{span}\{u_\theta, w\}$ bounded by $\mathbf{R}\nu_\theta$ which contains u_θ , and so is a singular direction of μ at q_θ by Lemma 5.3.

Step 3. We now apply Lemma 5.5 to the singular directions $v_{\theta,\varphi}$ uniformly in θ , producing a two-parameter family of points of Γ . Put

$$F(\theta, \varphi) := q_\theta + h v_{\theta,\varphi} = q + h((\cos \varphi - 1)u_\theta + \sin \varphi w),$$

so that $F(\theta, 0) = q$. Fix a closed interval $I \subset (0, \theta_1)$. We claim that there is $\varphi_0 \in (0, \pi)$ such that $F(\theta, \varphi) \in \Gamma$ for $\theta \in I$ and $|\varphi| < \varphi_0$. By Lemma 5.5, it suffices that $|\varphi| < \alpha(\theta)$, so that $v_{\theta,\varphi}$ is a singular direction of μ at q_θ not parallel to ν_θ , and that the plane orthogonal to $v_{\theta,\varphi}$ through $F(\theta, \varphi)$ meets $\text{int}(K)$. Both conditions hold when $\varphi = 0$: the first since $\alpha(\theta) > 0$, and the second since $F(\theta, 0) = q$ and $v_{\theta,0} = u_\theta \notin \mathbf{R}n$, so that the plane in question does not support Γ . Both conditions are open in (θ, φ) , since α is continuous and the second condition consists of strict inequalities between continuous functions. So the claim follows from the compactness of I .

Step 4. For each $0 < |\varphi| < \varphi_0$, the curve $\theta \mapsto F(\theta, \varphi)$, $\theta \in I$, lies in Γ and has velocity $h(\cos \varphi - 1)u'_\theta \neq 0$. Thus u'_θ is tangent to Γ at $F(\theta, \varphi)$, that is, $\langle \nu(F(\theta, \varphi)), u'_\theta \rangle = 0$. As $\varphi \rightarrow 0$, $F(\theta, \varphi) \rightarrow q$, so continuity of ν yields $\langle n, u'_\theta \rangle = 0$. But $\langle n, u'_\theta \rangle = -\sin \theta \neq 0$ for $\theta \in I$, which is the desired contradiction. $\square$

6. PROOF OF THEOREM 1.2

We prove Theorem 1.2 by showing that the slab-area property forces convexity, reducing the result to Theorem 1.1. Throughout this section let Γ be a $\mathcal{C}^3$ closed embedded surface satisfying the slab-area property for some $0 < h < \text{width}(\Gamma)$. Let K be the convex hull of Γ , ν be the outward unit normal of Γ , and $S = d\nu$ be the shape operator, with principal curvatures κ_1, κ_2 . We write $H := (\kappa_1 + \kappa_2)/2$ and $G := \kappa_1 \kappa_2$, for the mean and Gauss curvatures of Γ respectively. The support function p , the height functions ϕ_u , the level sets $\Gamma_{u,t}$, and the area density $R(u, t)$ are defined as in Section 3. Thus p is also the support function of K and $\text{width}(\Gamma) = \min_{u \in \mathbf{S}^2} \text{width}_u(\Gamma)$. Since the derivation of (3) used only the slab-area property and the coarea formula, it still holds for Γ whenever t and $t + h$ are regular values of ϕ_u with $-p(-u) < t < p(u) - h$.

The proof has two steps. First we use the singularities of the area density together with its h -periodicity to show that $H \leq 0$ on the portion of Γ lying sufficiently near its contact set with ∂K . We then apply the Hopf boundary point lemma to show that $\Gamma = \partial K$. This approach requires the following relation between the area density and Gauss curvature. By the standard genericity theorem for height functions, e.g., see [7, Lem. 5.3], there is an open dense set $\mathcal{G} \subset \mathbf{S}^2$ such that ϕ_u is Morse with pairwise distinct critical values for every $u \in \mathcal{G}$. The case of extrema in the next lemma is at the heart of Blaschke's converse to the Archimedes theorem [2, pp. 83, 161], and the general case appears in [27]. We include the short proof for completeness.

Lemma 6.1. *Let $u \in \mathcal{G}$, let y be a critical point of ϕ_u , and put $c := \phi_u(y)$. If y is a local minimum or maximum, then*

$$(8) \quad \lim_{\tau \downarrow 0} (R(u, c + \tau) - R(u, c - \tau)) = \begin{cases} \frac{2\pi}{\sqrt{G(y)}}, & \text{if } y \text{ is a local minimum,} \\ -\frac{2\pi}{\sqrt{G(y)}}, & \text{if } y \text{ is a local maximum.} \end{cases}$$

If y is a saddle point, then

$$(9) \quad \lim_{t \rightarrow c} \frac{R(u, t)}{\log(1/|t - c|)} = \frac{2}{\sqrt{|G(y)|}}.$$

Proof. Since the critical values are distinct, we may isolate y by a cutoff equal to 1 near y ; the contribution from the rest of the surface is continuous near c . By the Morse lemma there are coordinates $z = (z_1, z_2)$ centered at y in which

$$\phi_u = c + \frac{1}{2} (\varepsilon_1 z_1^2 + \varepsilon_2 z_2^2), \quad \varepsilon_i \in \{1, -1\}.$$

Writing the area form as $dA = \alpha(z) dz_1 dz_2$, we have $\alpha(0) = |G(y)|^{-1/2}$. At a local extremum, polar coordinates yield (8). At a saddle point, after an orthogonal change of coordinates we may write $\phi_u = c + z_1 z_2$. Since α is continuous, integration along $z_1 z_2 = t - c$ gives

$$R(u, t) = \left(\frac{2}{\sqrt{|G(y)|}} + o(1) \right) \log \frac{1}{|t - c|},$$

which proves (9). $\square$

Let $\mathcal{U} \subset \Gamma$ be the set of points $y \in \Gamma \cap \text{int}(K)$ such that the translate of $T_y \Gamma$ by $-h\nu(y)$ meets $\text{int}(K)$. Equivalently,

$$(10) \quad \mathcal{U} := \{y \in \Gamma : -p(-\nu(y)) < \langle y, \nu(y) \rangle - h < \langle y, \nu(y) \rangle < p(\nu(y))\}.$$

Lemma 6.2. $H \leq 0$ on $\mathcal{U}$.

Proof. The argument has two parts. First we show that $H \leq 0$ at points of $\mathcal{U}$ where $G \neq 0$. Then we rule out $H > 0$ at points where $G = 0$.

Part 1. Suppose first that $y \in \mathcal{U}$, $G(y) \neq 0$, and $u := \nu(y)$ belongs to $\mathcal{G}$. Put $c := \phi_u(y)$. By (10), $-p(-u) < c - h < c < p(u)$. By Lemma 6.1, $R(u, \cdot)$ has at c either a nonzero jump or a logarithmic singularity. Periodicity forces the same behavior at $c - h$, so $c - h$ is a critical value. Since the critical values are distinct, there is a unique critical point q at that height. Equations (8) and (9) show that y and q have the same Morse index and $|G(y)| = |G(q)|$.

Now vary u in a sufficiently small connected neighborhood $V \subset \mathcal{G}$. The point y extends to a $\mathcal{C}^1$ critical branch $y(v)$ with $\nu(y(v)) = v$, and periodicity pairs it with a single $\mathcal{C}^1$ critical branch $q(v)$ satisfying $\langle q(v), v \rangle = \langle y(v), v \rangle - h$. Differentiating in the direction $\xi \in T_v \mathbf{S}^2$, and using that $dy(v)[\xi]$ and $dq(v)[\xi]$ are orthogonal to v , gives $\langle y(v) - q(v), \xi \rangle = 0$. Hence

$$(11) \quad q(v) = y(v) - hv.$$

At $q(v)$ choose the unit normal v , whether or not it agrees with the outward normal there, and let $\tilde{S}_q$ be the corresponding shape operator. Differentiating (11) gives $dq = dy - h dv$ and $dv = S_y dy = \tilde{S}_q dq$. Thus $\tilde{S}_q(I - hS_y) = S_y$, and, after ordering the principal directions,

$$\tilde{\kappa}_i(q) = \frac{\kappa_i(y)}{1 - h\kappa_i(y)}, \quad i = 1, 2.$$

Since the Morse indices at y and q agree, S_y and $\tilde{S}_q$ have the same number of positive eigenvalues. A negative eigenvalue of S_y remains negative under the displayed transformation. Hence $1 - h\kappa_i(y) > 0$ for $i = 1, 2$. Since $|G(y)| = |G(q)|$, we have $(1 - h\kappa_1(y))(1 - h\kappa_2(y)) = 1$. Consequently

$$2 - 2hH(y) = (1 - h\kappa_1(y)) + (1 - h\kappa_2(y)) \geq 2,$$

so $H(y) \leq 0$. For general $y \in \mathcal{U}$ with $G(y) \neq 0$, the Gauss map is a local diffeomorphism near y . Approximate $\nu(y)$ by directions in $\mathcal{G}$ and pass to the limit.

Part 2. Suppose towards a contradiction that $X := \mathcal{U} \cap \{H > 0\}$ is nonempty. By Part 1, $G = 0$ on X . Thus the principal curvatures there are $\kappa = 2H > 0$ and 0. By a theorem of Hartman–Nirenberg [11, pp. 916–917], X is locally ruled by straight lines, and the tangent plane is constant along each ruling. Along a unit-speed ruling $\gamma(s) = y_0 + se$, the nonzero principal curvature has the form

$$(12) \quad \kappa(s) = \frac{\kappa_0}{1 + c_0 s}, \quad \kappa_0 > 0,$$

for some constant c_0 ; see [8, p. 419]. Let γ be a maximal such ruling in X . No endpoint of its maximal parameter interval can be finite. Indeed, if $s \rightarrow \beta$ at a finite endpoint, compactness of Γ gives a limit point $y_* \in \Gamma$. Boundedness of the principal curvatures implies $1 + c_0\beta \neq 0$, so κ has a positive finite limit and $H(y_*) > 0$. Along the ruling, ν and $\langle \gamma, \nu \rangle$ are constant, so (10) remains valid at y_* . Hence $y_* \in \mathcal{U}$, and thus $y_* \in X$. The unique local ruling through y_* then continues γ past y_* , contradicting maximality. Hence Γ contains an entire straight line, contrary to compactness. $\square$

Proposition 6.3. Γ is convex.

Proof. Let $C := \Gamma \cap \partial K$ be the contact set. At $y \in C$, a supporting plane of K is tangent to Γ , with outward normal $\nu(y)$. Hence $\langle y, \nu(y) \rangle = p(\nu(y))$. Set $d(y) := p(\nu(y)) - \langle y, \nu(y) \rangle$. Then $d = 0$ on C and $d > 0$ on $\Gamma \setminus C$. Put $\delta := \text{width}(\Gamma) - h > 0$. If $y \in \Gamma \setminus C$ is sufficiently close to C , then $0 < d(y) < \delta/2$ and

$$\langle y, \nu(y) \rangle - h + p(-\nu(y)) = \text{width}_{\nu(y)}(\Gamma) - h - d(y) \geq \delta - d(y) > 0.$$

Together with $d(y) > 0$, this gives (10). Hence $y \in \mathcal{U}$, and Lemma 6.2 gives $H \leq 0$ on $\Gamma \setminus C$ sufficiently close to C . Suppose that $\Gamma \setminus C$ is nonempty. Then C is a proper closed subset of Γ and has a relative boundary point y_0 . After a rigid motion we may assume that $y_0 = 0$ and $\nu(y_0) = (0, 0, 1)$, so that the supporting plane of K at y_0 is $\mathbf{R}^2 \times \{0\}$ and, near y_0 , Γ is the graph $\{(w, f(w)) : w \in W\}$ of a $\mathcal{C}^3$ function

f on a neighborhood W of 0 in $\mathbf{R}^2$. We choose W so small that $H \leq 0$ at every noncontact point. Then we have

$$\operatorname{div} \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} = -2H \geq 0$$

on the noncontact region. Choose a noncontact point w_1 sufficiently close to the contact set, and let r be its distance to that set, with $B_r(w_1) \subset W$. Then $B_r(w_1)$ contains no contact point and its boundary contains one, say w_0 . Write the supporting plane of K at $(w_0, f(w_0))$ as $z = \ell(w)$. Since Γ lies below this plane, $v := f - \ell$ satisfies $v < 0$ in $B_r(w_1)$, $v(w_0) = 0$, $\nabla v(w_0) = 0$. Since ℓ is affine, the mean-curvature inequality becomes $a^{ij}(\nabla f)v_{ij} \geq 0$ on the disk, where

$$a^{ij}(\xi) = \frac{\delta_{ij}}{\sqrt{1 + |\xi|^2}} - \frac{\xi_i \xi_j}{(1 + |\xi|^2)^{3/2}}.$$

This is uniformly elliptic on $B_r(w_1)$. The Hopf boundary point lemma gives a strictly positive outward normal derivative of v at w_0 , contradicting $\nabla v(w_0) = 0$. Hence $\Gamma \subset \partial K$. Finally, Γ is a closed surface contained in the topological sphere ∂K , so it is open in ∂K by invariance of domain, and closed. Hence $\Gamma = \partial K$. $\square$

Note 6.4. The $\mathcal{C}^3$ hypothesis in Theorem 1.2 is used only in the proof of Lemma 6.2, where the Codazzi equations are applied to establish (12). That formula, and hence Theorem 1.2, extends to $\mathcal{C}^2$ surfaces, but we do not pursue this here.

Note 6.5. Proof of Theorem 1.2 applies more generally for immersed surfaces whose self-intersection set is nowhere dense. Indeed, the arguments of this section are local and apply with either choice of a local unit normal ν . In the proof of Proposition 6.3, ν is chosen near a relative boundary point of the contact set to agree with the outward normal of K there. The hypothesis on the self-intersections ensures that generic height functions have distinct critical values, which is used in Lemmas 6.1 and 6.2, since two sheets of Γ whose critical values agree for an open set of directions coincide on an open set. In the last step, invariance of domain shows that the immersion is a covering map of ∂K , and hence an embedding.

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