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SOLUTION TO THE SHADOW PROBLEM IN 3-SPACE

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ABSTRACT. If a convex surface, such as an egg shell, is illuminated from any given direction, then the corresponding shadow cast on the surface forms a connected subset. The shadow problem, first studied by H. Wente in 1978, asks whether a converse of this phenomenon is true as well. In this report it is shown that the answer is yes provided that each shadow is *simply* connected; otherwise, the answer is no. Further, the motivations behind this problem, and some ramifications of its solution for studying constant mean curvature surfaces in 3-space (soap bubbles) are discussed.

1. INTRODUCTION

Let $M \subset \mathbf{R}^3$ be a smooth *convex surface*, i.e., the boundary of a convex body; let $n\colon M \to \mathbf{S}^2$ denote the outward unit normal vectorfield, which we also refer to as the *Gauss map*, of M; and let $u \in \mathbf{S}^2$ be a unit vector. Suppose that M is illuminated by parallel rays of light flowing in the direction of u, see Figure 1. Then, the *shadow*

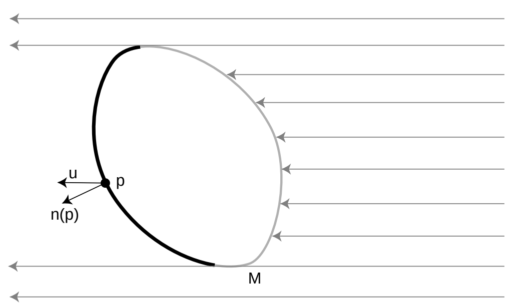

FIGURE 1. The shadow cast on a surface M, when illuminated by light rays parallel to u corresponds to points p in M where the inner product between u and the unit normal $n(p)$ is positive.

cast on M, i.e., the set of points in M not reached by the rays of light, is given by

$$S_u := \{\, p \in M \mid \langle n(p), u \rangle > 0 \,\}, \tag{1.1}$$

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where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^3$. It is intuitively clear, and not too difficult to show [6], that if M is convex, then, for every $u \in \mathbf{S}^2$, S_u is a connected subset of M.

It is natural to ask whether the connectedness of the shadows characterizes convex surfaces, i.e., whether the converse of the above phenomenon holds as well. More precisely, let M be a *closed* (i.e., compact and connected) surface immersed in $\mathbf{R}^3$. Suppose that M is oriented, so that the Gauss map is globally well-defined. Then, for every unit vector $u \in \mathbf{S}^2$, let the corresponding shadow, S_u, be defined as in (1.1). Suppose that for every u, S_u is a connected subset of M. Does it then follow that M is convex?

In 1978, motivated by problems concerning the stability of constant mean curvature surfaces, H. Wente appears to have been the first person to have studied the above question [18], see Section 4, which has since become known as the *shadow problem* (a.k.a. the *illumination conjecture*). Recently, the author has proved that this problem has a positive solution provided that the shadows are *simply* connected:

Theorem 1.1. *Let M be an oriented compact surface immersed in $\mathbf{R}^3$. Suppose that for every $u \in \mathbf{S}^2$, the corresponding shadow, S_u, is* simply *connected. Then M is convex. In particular, M is embedded and homeomorphic to $\mathbf{S}^2$.*

A proof of the above theorem is outlined in Section 2. Furthermore, in Section 3, we will show that the additional condition in Theorem 1.1 (the word *simply*) is in fact necessary, as there exist embedded closed surfaces of genus one all of whose shadows are connected. Some ramifications for studying constant mean curvature surfaces will be discussed in Section 4.

Note 1.2. If M is assumed to be *simply* connected, then the assumption, in Theorem 1.1, that the shadows be 'simply connected' may be weakened to 'connected' [6].

Note 1.3. The compactness assumption in Theorem 1.1 cannot be removed. Suppose, for instance, that M is a hyperbolic paraboloid, such as the one given by the graph of the equation $z = xy$. Then all the shadows of M are simply connected, even though M is not a convex surface. To see this, let H_u denote the (open) hemisphere in $\mathbf{S}^2$ determined by the unit vector u, i.e., let $H_u := \{x \in \mathbf{S}^2 \mid \langle x, u \rangle > 0\}$. A direct computation shows that the Gauss map of M, n, is a homeomorphism into $H_{(0,0,1)}$ (the Northern hemisphere). Further, note that $S_u = n^{-1}(H_u) = n^{-1}(H_u \cap H_{(0,0,1)})$. Thus, since $H_u \cap H_{(0,0,1)}$ is simply connected, it follows that S_u is simply connected as well.

2. Outline of the Proof

The proof is by contradiction, and is organized into three steps described below. The first two steps employ techniques from Morse theory [14], and the third step, which is the main part of the proof, introduces a topological invariant for shadows by permuting the critical points of height functions. For a full treatment of all the details, we refer the reader to [10] or [6].

2.1. **Critical points of height functions.** Suppose that M satisfies the hypothesis of Theorem 1.1, but is not convex. Then there exists a unit vector $v \in \mathbf{S}^2$ such that the corresponding height function $h_v \colon M \to \mathbf{R}$, defined by

$$h_v(p) := \langle p, v \rangle,$$

has at least three nondegenerate critical points, see Figure 2. This follows from basics

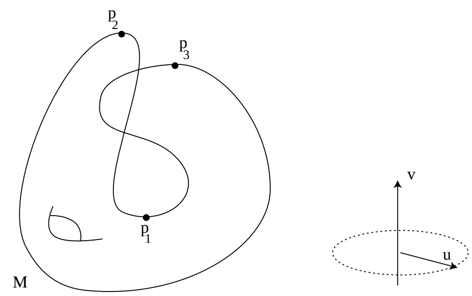

FIGURE 2. If a closed surface, M, immersed in $\mathbf{R}^3$, is not convex, then there exists a direction, v, such that the corresponding height function, h_v, is a Morse function with at least three critical points.

of the theory of *tight immersions* [4] going back to the works of Chern and Lashof [5]: let $\#C(h_u)$ denote the number of critical points of a Morse height function h_u, and let K be the Gauss curvature of M; then one has the following formula

$$\frac{1}{2}\int_{\mathbf{S}^2} \#C(h_u)\, du = \int_M |K(x)|\, dx.$$

Note that the integral on the left is well-defined, because for almost every $u \in S^2$, h_u is a Morse function (this is an easy application of Sard's theorem), and consequently $\#C(h_u)$ is finite. The integral on the right is known as the *total absolute curvature*, which is bounded below by 4π, because the Gauss map is surjective. $\int_M |K(x)|dx$ attains its minimum only when M is convex [5, Thm 3]. Thus, assuming that M is not convex, $\int_M |K(x)|dx > 4\pi$. Consequently, by the above formula, there has to exist a Morse function, h_v, with more than two critical points.

2.2. **Regularity of the boundary of shadows.** Let $v^\perp := \{u \in \mathbf{S}^2 \mid \langle u, v \rangle = 0\}$. Using Sard's theorem, it can be shown that, after a perturbation of v, we can assume that there exists a vector $u_0 \in v^\perp$, such that the boundary of the corresponding shadow, ∂S_{u_0}, is a regular submanifold of M. This is a consequence of the fact that, for almost every $u \in \mathbf{S}^2$, ∂S_u is regular, which, briefly, may be proved as follows: define the *shadow function* $f_u \colon M \to \mathbf{R}$ by

$$f_u(p) := \langle n(p), u \rangle,$$

and observe that $\partial S_u \subset f_u^{-1}(0)$. Further, let UTM denote the unit tangent bundle of M, i.e., $UTM := \{(p, t_p) \mid p \in M,\ t_p \in T_pM, \text{ and } \|t_p\| = 1\}$. Define $\tau \colon UTM \to \mathbf{S}^2$

and $\pi\colon UTM \to M$, by $\tau(p,t_p) := t_p$ and $\pi(p,t_p) := p$ respectively.

$$\begin{array}{ccc} UTM & \xrightarrow{\tau} & \mathbf{S}^2 \\ \big\downarrow \pi & & \\ M & & \end{array}$$

Then $f_u^{-1}(0) = \pi(\tau^{-1}(u))$. Let u be a regular value of τ. Then $\tau^{-1}(u)$ is a regular curve in UTM. Further, it is not too difficult to show that π is an embedding on $\tau^{-1}(u)$. Hence, by Sard's theorem, for almost every u, $f_u^{-1}(0)$, and consequently ∂S_u, is a regular curve (for more results of this type and an introduction to studying geometry of the shadow boundaries on illuminated surfaces see [11]).

After a rotation of the coordinate axis, and for the sake of convenience, we assume from now on that $v = (0,0,1)$ and $u_0 = (1,0,0)$. Further, we parameterize $v^{\perp}$ by $u(\theta) := (\cos\theta, \sin\theta, 0)$, $\theta \in [0, 2\pi]$.

2.3. **Induced permutations on the critical points.** Let p_i, $i = 1,2,3$, denote three critical points of h_v. For every $\theta \in [0,2\pi]$, we define a permutation $\sigma(\theta) \in \mathrm{Sym}(p_1,p_2,p_3)$, the symmetric group of three elements, as follows.

Fix $\theta \in [0,2\pi]$. Note that $p_i \in \partial S_{u(\theta)}$; furthermore, since p_i is a *nondegenerate* critical point of the height function h_v, it follows that $\partial S_{u(\theta)}$ is regular in a neighborhood of p_i, see Figure 3. This together with the simply-connectedness of $S_{u(\theta)}$

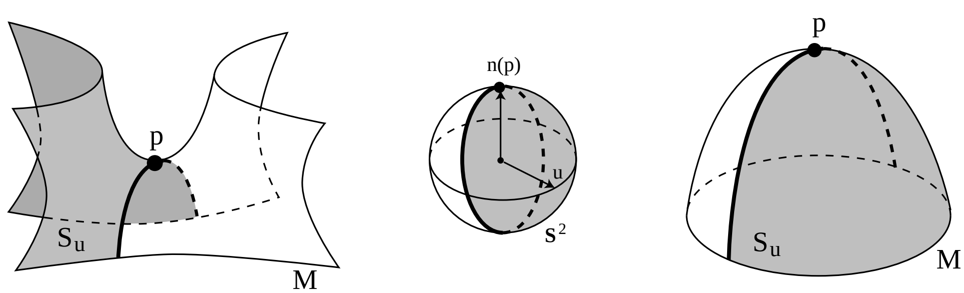

FIGURE 3. If p is a regular critical point of the height function h_v, then $n(p) = \pm v$, and for every $u \in \mathbf{S}^2$ orthogonal to $n(p)$ the boundary of the corresponding shadow ∂S_u is regular in a neighborhood of p; because, n is a local diffeomorphism at p, and ∂S_u is the pull-back via n of a great circle in $\mathbf{S}^2$.

implies that there exists a simple closed curve T in the closure of $S_{u(\theta)}$ such that: (i) T is composed of three smooth arcs which end at p_i, (ii) each arc meets $\partial S_{u(\theta)}$ transversally, and (iii) the interior of each arc lies in $S_{u(\theta)}$. We say that such a curve is a *standard triangle* for $S_{u(\theta)}$, see Figure 4. Since $S_{u(\theta)}$ is simply connected, T bounds a unique region in $S_{u(\theta)}$. This region inherits an orientation from M (recall that M is, by assumption, oriented), which in turn induces a preferred sense of direction on T. The induced direction on T determines a permutations for p_i in a natural way; for instance, suppose that as we move along T away from p_1 we encounter p_2

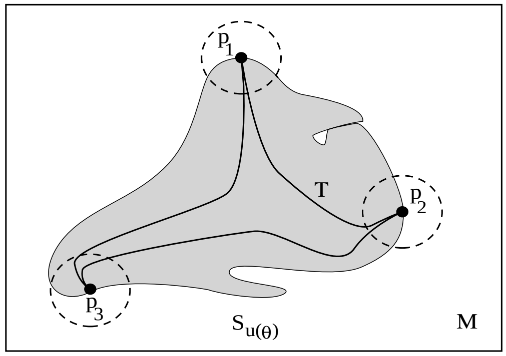

FIGURE 4. Each shadow, $S_{u(\theta)}$, contains a *standard triangle*. Note that the boundary of the shadow is a regular curve in a neighborhood of the critical points p_i.

before reaching p_3, then we say that the induced permutation is the cycle $(p_1p_2p_3)$. Finally, note that the induced permutation on p_i does not depend on the choice of the standard triangle; because, if T' is any other standard triangle in $S_{u(\theta)}$, then T' and T are homotopic in $S_{u(\theta)}$ by the simply-connectedness of $S_{u(\theta)}$. So we conclude that each shadow $S_{u(\theta)}$ determines a unique permutation on $\{p_1, p_2, p_3\}$, which we denote by $\sigma(\theta)$.

We claim that the map $\sigma\colon [0, 2\pi] \to \mathrm{Sym}(p_1, p_2, p_3)$ which we defined above is constant. To this end, since $[0, 2\pi]$ is connected, it suffices to show that σ is locally constant. This follows from the fact that whenever θ and θ' are sufficiently close, then $S_{u(\theta)}$ and $S_{u(\theta')}$ have a standard triangle in common, see Figure 5. The proof

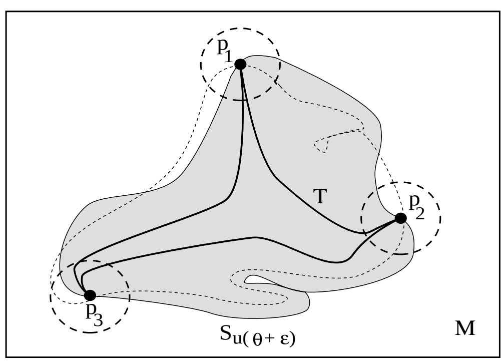

FIGURE 5. For every θ there exists an $\epsilon > 0$ such that the shadows $S_{u(\theta)}$ and $S_{u(\theta+\epsilon)}$ have a standard triangle in common. This shows that the induced permutation on $\{p_1, p_2, p_3\}$ is locally constant.

of this is based on the compactness of T, the assumption that T meets $\partial S_{u(\theta)}$ only at p_i and does so transversally, and the observation that in a neighborhood of p_i $\partial S_{u(\theta)}$ depends continuously on θ.

On the other hand, it is not difficult to show that $\sigma(0) \neq \sigma(\pi)$. To see this, recall that $\partial S_{u(0)}$ is regular by construction. This implies that $\partial S_{u(\pi)} = \partial S_{u(0)}$. Further, since $S_{u(0)}$ is, by assumption, simply connected, $\partial S_{u(0)}$ is connected. In particular, $\partial S_{u(0)}$ is a simple closed curve passing through p_i. Suppose that $\partial S_{u(0)}$ is given the orientation induced by $S_u(0)$, and note that the corresponding permutation induced on p_i coincides with $\sigma(0)$, because all standard triangles in $S_{u(0)}$ are homotopic to $\partial S_{u(0)}$. Similarly, if $\partial S_{u(\theta)}$ is oriented by $S_{u(\pi)}$, then this gives rise to a permutation of p_i which is identical with $\sigma(\pi)$. $S_{u(0)}$ and $S_{u(\pi)}$ induce opposite orientations on $\partial S_{u(\theta)}$. Hence $\sigma(0) = -\sigma(\pi)$, which produces the desired contradiction and completes the proof.

3. A Counterexample

In this section we show that Theorem 1.1 does not remain valid if the condition that the shadows be 'simply connected' is replaced by 'connected'. More specifically, we show that there exists a smooth embedded surface of genus one all of whose shadows are connected. This surface is given by building a tube around a closed curve without any pairs of parallel tangent lines. An explicit example of such a curve, formulated by Ralph Howard, is given by $\gamma(t) := (x(t), y(t), z(t))$, where

$$\begin{aligned} x(t) &:= -\cos(t) - \frac{1}{20}\cos(4t) + \frac{1}{10}\cos(2t), \\ y(t) &:= +\sin(t) + \frac{1}{10}\sin(2t) + \frac{1}{20}\sin(4t), \\ z(t) &:= -\frac{46}{75}\sin(3t) - \frac{2}{15}\cos(3t)\sin(3t), \end{aligned} \tag{3.1}$$

$t \in [0, 2\pi]$. Figure 6 shows the pictures of a small tube built around the above curve. Let Γ denote the trace of γ. Since Γ is a regular submanifold, it follows from the

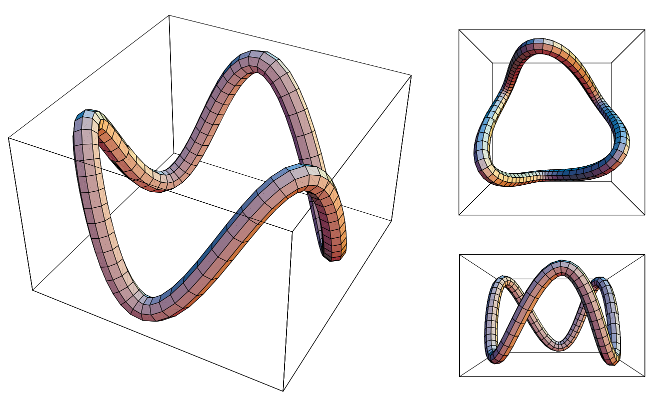

Figure 6. Three different views of a nonconvex surface all of whose shadows are connected. This surface is constructed by building a tube around a curve with no pair of parallel tangents.

tubular neighborhood theorem that there exists an $r > 0$ such that

$$M := \{x \in \mathbf{R}^3 \mid \operatorname{dist}(x, \Gamma) = r\}$$

is a smooth surface, where $\operatorname{dist}(x, \Gamma) := \inf_{y \in \Gamma} \|x - y\|$. We claim that, since Γ has no pair of parallel tangent lines, each shadow of M is a connected subset. Before proving this, however, we describe a general procedure for constructing Γ.

Let $T \subset \mathbf{S}^2$ be a smooth simple closed curve such that (i) the origin is contained in the interior of the convex hull of T, $(0,0,0) \in \operatorname{int}\operatorname{conv} T$, and (ii) T does not contain any pair of antipodal points. Although it is not immediately clear that such curves exist, they are not difficult to construct. Figure 7 shows an example, which is perhaps, qualitatively speaking, the simplest. Let $T(s)$, $s \in \mathbf{R}$, denote a periodic

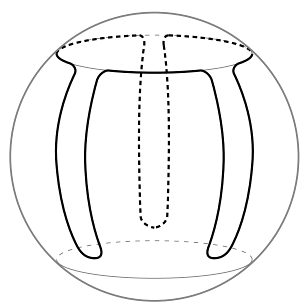

FIGURE 7. A simple closed curve on the sphere which contains the origin in the interior of its convex and is disjoint from its antipodal reflection. An appropriate integration of the above yields a space curve with no parallel tangents.

parameterization of T by arclength. So, assuming T has total length L, we have $T(s + L) = T(s)$. Since $(0,0,0) \in \operatorname{int}\operatorname{conv} T$, there exists a (density) function $v(s)$ with period L such that $\int_0^L v(s)T(s)\,ds = 0$; or, intuitively speaking, it is possible to distribute mass along T so that the center of gravity of the resulting object coincides with the origin. Now set

$$\gamma(t) := \int_0^t v(s)T(s)\,ds.$$

Then $\gamma(t+L) = \gamma(t)$. Further, $\gamma'(t)/\|\gamma'(t)\| = T(t)$. Thus γ is a closed curve whose tangential spherical image coincides with T. In particular, γ has no parallel tangent lines. Hence Γ (the trace of γ) is the desired curve.

Next we show that M, given by a small tube around Γ, has connected shadows. To see this, let $\pi\colon M \to \Gamma$ be the obvious projection, i.e., the nearest point map. For every $x \in \Gamma$, let $F_x := \pi^{-1}(x)$ be the corresponding fiber. Note that (i) each fiber, F_x, is a circle, (ii) the image of each fiber under the Gauss map, $n(F_x)$, is the great circle in $\mathbf{S}^2$ which lies in the plane perpendicular to $T(x)$, and (iii) n is one-to-one on each F_x. Let $u \in S^2$, and let S_u be the corresponding shadow cast on M. Recall that $S_u = n^{-1}(H_u)$, where $H_u := \{x \in S^2 \mid \langle x, u\rangle > 0\}$ is an open hemisphere, see Figure 8. Thus, for each fiber, F_x, we have only two possibilities:

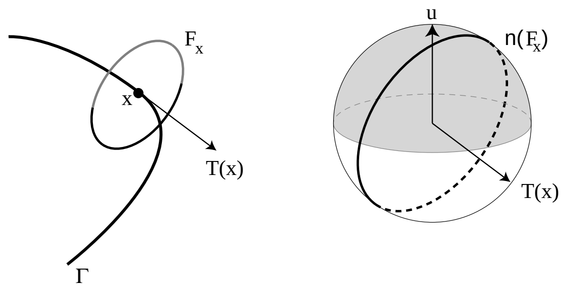

FIGURE 8. Unless $T(x)$ and u are parallel, the fiber F_x of the tube around Γ intersects the shadow S_u along an open semicircle.

either F_x intersects S_u in an open half-circle, or F_x is disjoint from S_u. But, by construction of Γ, the latter occurs for at most for one $x \in \Gamma$. Hence, it follows that each shadow, S_u, is either homeomorphic to a disk or an annulus. In particular, S_u is connected for every $u \in \mathbf{S}^2$.

Note 3.1. It is an elementary and well-known fact that if Γ is a closed curve, then its tangential spherical image (a.k.a. tangent indicatrix or *tantrix*), contains the origin in the interior of its convex hull. Here we showed that a converse of this phenomenon holds as well. This observation is also known, and has been attributed to Löwner; but it is not clear if it had ever been published by him. See [12] for detailed proofs and historical comments. A proof may also be found in [8, p. 168]

Note 3.2. It is possible to construct a simple closed curve without parallel tangents which lies on a cylinder with a convex base. In fact, the equations (3.1) give one such example. So a loop without parallel tangents may lie on the boundary of a convex body. Interestingly enough, however, no such curve may be constructed on an ellipsoid. This follows from recent results of Joel Weiner [21] or Bruce Solomon [17] who showed that the tantrix of a spherical curve, if embedded, divides the sphere into equal areas. Consequently, any loop on a sphere has to have a pair of parallel tangents. Further, ellipsoids must have this property as well, because they are equivalent to the sphere up to a linear transformation. It would be interesting to know if ellipsoids are the only closed surfaces which admit no loops without parallel tangent[1].

4. APPLICATIONS

4.1. **Stable Constant Mean Curvature Surfaces.** In this section we discuss the original motivation for studying the shadow problem, and indicate how one can obtain a classical isoperimetric result using Theorem 1.1.

Let M be an oriented, closed, and stable constant mean curvature (CMC) surface immersed in $\mathbf{R}^3$. Stable means that M is a critical surface for the area functional

[1] Note added in proof: since this paper was first written, the author and Bruce Solomon have proved that the property of having no loops without parallel tangent lines (*skew loops*), does indeed characterize ellipsoids amongst all closed surfaces immersed in 3-space [7].

subject to a volume constraint. In 1978, when the shadow problem seems to have first originated, it was not yet known that M is necessarily a (round) sphere. Motivated by this question, one might make the following observation: M, much like a sphere, has connected shadows. This is based on a variation argument, described below, which the author first learned from Henry Wente [18].

For all $u \in S^2$, the shadow function $f_u \colon M \to \mathbf{R}$, $f_u(p) := \langle n(p), u\rangle$, is a Jacobi field on M, i.e., for the perturbation

$$p \longmapsto p + t f_u(p) n(p),$$

the first variation of volume and the first and second variation of area are all zero; because, the variations corresponds to a rigid motion of M in the direction u. Consider the nodal regions of f_u on M. These are the sets where f_u is either positive or negative, and correspond, therefore, to the shadows S_u and S_{-u}, respectively. Suppose, towards a contradiction, that S_u is not connected, then there are at least three nodal regions A_i, $i = 1, 2, 3$. Consequently, one can form three functions f_i by setting $f_i := f$ on A_i and $f_i := 0$ elsewhere. One can then take a suitable linear combination $\sum_{i=1}^{3} \lambda_i f_i$, to obtain a function for which the first variation of volume is zero but the second variation of area is negative, contradicting the stability assumption. Hence, we conclude that all shadows of M are connected.

Suppose now that M is simply connected, then, see Note 1.2, the connectedness of the shadows of M imply that each shadow is simply connected. Hence, by Theorem 1.1, it follows that M is convex. In particular, M is embedded. Consequently, by applying the maximum principle together with the reflection technique introduced by Aleksandrov [1], it follows that M is a sphere.

The above result is well-known, and may be regarded as a weak version of a theorem of Hopf [9, p. 138], or a theorem of Barbosa and do Carmo [2]. Hopf showed, without assuming stability, that any closed CMC surface of genus zero must be a sphere, and Barbosa and do Carmo proved that a closed oriented surface of higher genus must also be a sphere provided it is stable (for an elementary proof of this result, see [19]). Finally, Wente showed that the stability assumption in higher genus is not superfluous [20] by constructing a CMC torus in $\mathbf{R}^3$; thus, settling a famous and long standing question of Hopf [9, Pg. 131].

In closing this section, we should also point out that a number of results concerning the connection between the number of components of nodal regions of the shadow function (the *vision number*) and the stability index of complete minimal surfaces in $\mathbf{R}^3$ have been obtained by Jaigyoung Choe [3].

4.2. **Convexity of the level sets of H-graphs.** Recently, shadows on illuminated surfaces have been studied within the context of another problem involving constant mean curvature. This problem, unlike those mentioned in the previous subsection, is still open. Let $\Omega \subset \mathbf{R}^2$ be a convex domain, and $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution to the following boundary value problem:

$$\mathrm{Div}\Big(\frac{\mathrm{grad}\, f}{\sqrt{1 + \|\,\mathrm{grad}\, f\|^2}}\Big) = 2H \quad \text{on } \Omega, \text{ and } f = 0 \quad \text{on } \partial\Omega.$$

Let M denote the graph of f. Then M has constant mean curvature H. Intuitively, one may think of M as the membrane of least area, spanned by $\partial\Omega$, which traps a given volume above the xy-plane. It has been a well-known and long standing problem [13] to show that the level sets of M, and those given by equations of similar type, are convex. Recently, John McCuan [15] has obtained a number of results on this problem. In particular, he has shown that for every unit vector $u(\theta) := (\cos\theta, \sin\theta, 0)$, the set $X_{u(\theta)} := \{x \in \overline{\Omega} \mid \langle \operatorname{grad} f(x), u(\theta)\rangle = 0\}$ is a connected regular curve, assuming that $\partial\Omega$ has strictly positive curvature. This implies that the shadow $S_{u(\theta)}$ is a simply connected subset of M, because $X_{u(\theta)}$ is the projection of $\partial S_{u(\theta)}$ into the xy-plane. One is then led to consider the following question [16]: does the simply-connectedness of the shadows $S_{u(\theta)}$ imply that the level sets of M are convex? The answer is negative, see Figure 9. At present it is not clear what

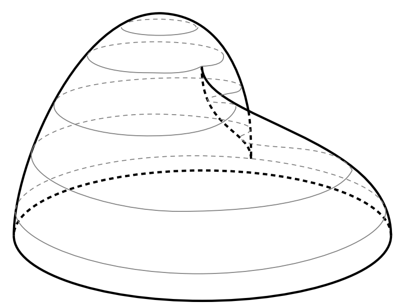

FIGURE 9. A graph with zero boundary values over a convex domain which has nonconvex level sets, even though the shadows $S_{u(\theta)}$ are simply connected. $S_{u(\theta)}$ is simply connected because the graph has a unique critical point.

shadow property, if any, would characterize the convexity of level sets.

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