Math 497C
Curves and Surfaces
Fall 2004, PSU

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Lecture Notes 15

2.13 The Geodesic Curvature

Let $\alpha\colon I \to M$ be a unit speed curve lying on a surface $M \subset \mathbf{R}^3$. Then the *absolute geodesic curvature* of α is defined as

$$|\kappa_g| := \big\|(\alpha'')^\top\big\| = \big\|\alpha'' - \big\langle\alpha'', n(\alpha)\big\rangle n(\alpha)\big\|,$$

where n is a local Gauss map of M in a neighborhood of $\alpha(t)$. In particular note that if $M = \mathbf{R}^2$, then $|\kappa_g| = \kappa$, i.e., absolute geodesic curvature of a curve on a surface is a generalization of the curvature of curves in the plane.

Exercise 1. Show that the absolute geodesic curvature of great circles in a sphere and helices on a cylinder are everywhere zero.

Similarly, the *(signed) geodesic curvature* generalizes the notion of the signed curvature of planar curves and may be defined as follows.

We say that a surface $M \subset \mathbf{R}^3$ is *orientable* provided that there exists a (global) Gauss map $n\colon M \to \mathbf{S}^2$, i.e., a *continuous* mapping which satisfies $n(p) \in T_pM$, for all $p \in M$. Note that if n is a global Gauss map, then so is $-n$. In particular, any orientable surface admits precisely two choices for its global Gauss map. Once we choose a Gauss map n for an orientable surface, then M is said to be *oriented*.

If M is an oriented surface (with global Gauss map n), then, for every $p \in M$, we define a mapping $J\colon T_pM \to T_pM$ by

$$JV := n \times V.$$

Exercise 2. Show that if $M = \mathbf{R}^2$, and $n = (0, 0, 1)$, then J is counter clockwise rotation about the origin by $\pi/2$.

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Then the *geodesic curvature* of a unit speed curve $\alpha \colon I \to M$ is given by

$$\kappa_g := \big\langle \alpha'', J\alpha' \big\rangle.$$

Note that, since $J\alpha'$ is tangent to M,

$$\big\langle \alpha'', J\alpha' \big\rangle = \big\langle (\alpha'')^\top, J\alpha' \big\rangle.$$

Further, since $\|\alpha'\| = 1$, α'' is orthogonal to α', which in turn yields that the projection of α'' into the tangent plane is either parallel or antiparallel to $J\alpha'$. Thus $\kappa_g > 0$ when the projection of α'' is parallel to $J\alpha'$ and is negative otherwise.

Note that if the curvature of α does not vanish (so that the principal normal N is well defined), then

$$\kappa_g = \kappa \big\langle N, JT \big\rangle. \tag{1}$$

Exercise 3. Let $\mathbf{S}^2$ be oriented by its outward unit normal, i.e., $n(p) = p$, and compute the geodesic curvature of the circles in $\mathbf{S}^2$ which lie in planes $z = h$, $-1 < h < 1$. Assume that all these circles are oriented consistently with respect to the rotation about the z-axis.

Next we derive an expression for κ_g which does not require that α have unit speed. To this end, let $s \colon I \to [0, L]$ be the arclength function of α, and recall that $\overline{\alpha} := \alpha \circ s^{-1} \colon [0, L] \to M$ has unit speed. Thus

$$\kappa_g = \overline{\kappa}_g(s) = \big\langle \overline{\alpha}''(s), J\overline{\alpha}'(s) \big\rangle.$$

Now recall that $(s^{-1})' = 1/\|\alpha'\|$. Thus by chain rule.

$$\overline{\alpha}'(t) = \alpha'\big(s^{-1}(t)\big) \cdot \frac{1}{\|\alpha'(s^{-1}(t))\|}.$$

Further, differentiating both sides of the above equation yields

$$\overline{\alpha}'' = \alpha''\big(s^{-1}\big) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'\big(s^{-1}\big) \cdot \frac{-\Big\langle \alpha''\big(s^{-1}\big), \alpha'\big(s^{-1}\big) \Big\rangle}{\|\alpha'(s^{-1})\|^3}.$$

Substituting these values into the last expression for $\overline{\kappa}_g$ above yield

$$\kappa_g = \frac{\big\langle \alpha'', J\alpha' \big\rangle}{\|\alpha'\|^3}. \tag{2}$$

Exercise 4. Show that the equations (1) and (2) above are equivalent. Conclude that κ_g is invariant under reparametrization.

Next we show that the geodesic curvature is intrinsic, i.e., it is invariant under isometries of the surface. To this end define $\tilde{\alpha}' \colon \alpha(I) \to \mathbf{R}^3$ be the vectorfield along $\alpha(I)$ given by

$$\tilde{\alpha}'\big(\alpha(t)\big) = \alpha'(t).$$

Then one may immediately check that

$$\alpha''(t) = \overline{\nabla}_{\alpha'(t)}\tilde{\alpha}'.$$

Thus

$$\big\langle \alpha'', J\alpha' \big\rangle = \big\langle (\alpha'')^{\top}, J\alpha' \big\rangle = \big\langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \big\rangle.$$

and it follows that

$$\kappa_g = \frac{\big\langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \big\rangle}{\|\alpha'\|^3}.$$

We say that a curve $\alpha \colon I \to M$ is a *geodesic* provided that its geodesic curvature $\kappa_g \equiv 0$.

Exercise 5. Show that if α is parametrized by arclength, then

$$|\kappa_g| = \|\nabla_{\alpha'}\tilde{\alpha}'\|.$$

Conclude that α is a geodesic if and only if $\nabla_{\alpha'}\tilde{\alpha}' \equiv 0$.

Exercise 6. Show that if α is a geodesic, then it must have constant speed.

Now recall that ∇ is intrinsic, which immediately implies that $|\kappa_g|$ is intrinsic by the last exercise. Thus to complete the proof that κ_g is intrinsic it remains to show that J is intrinsic. To see this let $X \colon U \to M$ be a local patch, then

$$JX_i = \sum_{j=1}^{2} b_{ij}X_j.$$

In particular,

$$JX_1 = b_{11}X_1 + b_{12}X_2.$$

Now note that

$$0 = \langle JX_1, X_1 \rangle = b_{11}g_{11} + b_{12}g_{21}.$$

Further,

$$g_{11} = \langle X_1, X_1 \rangle = \langle JX_1, JX_1 \rangle = b_{11}^2 g_{11} + 2b_{11}b_{12}g_{12} + b_{12}^2 g_{22}.$$

Solving for b_{21} in the next to last equation, and substituting in the last equation yields

$$g_{11} = b_{11}^2 g_{11} - 2b_{11}^2 g_{11} + b_{11}^2 \frac{g_{11}^2}{g_{21}^2} g_{22} = b_{11}^2(-g_{11} + \frac{g_{11}^2}{g_{21}^2} g_{22}).$$

Thus b_{11} may be computed in terms of g_{ij} which in turn yields that b_{12} may be computed in terms of g_{ij} as well. So JX_1 nay be expressed intrinsically. Similarly, JX_2 may be expressed intrinsically as well. So we conclude that J is intrinsic.

2.14 Geodesics in Local Coordinates

Here we will derive a system of ordinary differential equations, in terms of any local coordinates, whose solutions yield geodesics.

To this end let $X\colon U \to M$ be a patch, and $\alpha\colon I \to X(U)$ be a unit speed one-to-one curve. Then we may write

$$X(u(t)) = \alpha(t),$$

by letting $u(t) := X^{-1}(\alpha(t))$. Next note that, if u_i denotes the coordinates of u, i.e., $u(t) = (u_1(t), u_2(t))$, then by the chain rule,

$$\alpha' = \sum_{i=1}^{2} X_i u_i',$$

which in turn yields

$$\alpha'' = \sum_{i,j=1}^{2} X_{ij} u_i' u_j' + X_i u_i'' = \sum_{i,j,k=1}^{2} (\Gamma_{ij}^k X_k + \ell_{ij} N) u_i' u_j' + X_i u_i'',$$

by Gauss's formula. Consequently,

$$(\alpha'')^\top = \sum_{i,j,k=1}^{2} (\Gamma_{ij}^k X_k) u_i' u_j' + X_i u_i'' = \sum_{i,j,k=1}^{2} (\Gamma_{ij}^k u_i' u_j' + u_k'') X_k.$$

So, since $|\kappa_g| = \|(\alpha'')^\top\|$, we conclude that α is a geodesic if and only if

$$\sum_{i,j=1}^{2} (\Gamma_{ij}^k u_i' u_j' + u_k'') = 0$$

for $k = 1, 2$. In other words, for α to be a geodesic the following two equations must be satisfied:

$$u_1'' + \Gamma_{11}^1 (u_1')^2 + 2\Gamma_{12}^1 u_1' u_2' + \Gamma_{22}^1 (u_2')^2 = 0$$
$$u_2'' + \Gamma_{11}^2 (u_1')^2 + 2\Gamma_{12}^2 u_1' u_2' + \Gamma_{22}^2 (u_2')^2 = 0$$

Exercise 7. Write down the equations of the geodesic in a surface of revolution. In particular, verify that the great circles in a sphere are geodesics.

2.15 Parallel Translation

Here we will give another interpretation for the concept of geodesic curvature. Let $\alpha \colon I \to M$ be a simple curve and V be a vector field on M. We say that V is *parallel* along α provided that

$$\nabla_{\alpha'(t)} V = \big((V \circ \alpha)'\big)^\top = 0.$$

for all $t \in I$. Recall that α is a geodesic if and only if its velocity is parallel (i.e., $\nabla_{\alpha'(t)} \tilde{\alpha}' \equiv 0$)

Exercise 8. Show that if V is parallel along α, then its length is constant.

Exercise 9. Show that if V and W are a pair of parallel vector fields along α, then the angle between them is constant.

Proposition 10. *If α is a unit speed curve on a surface, and V is a parallel vector field along α, which makes an angle ϕ with the tangent vector of α, then $\kappa_g = \phi'$.*

Proof. We may assume that V has unit length. Then we may write:

$$T = \cos(\phi) V + \sin(\phi) JV.$$

Since $\nabla_T V = 0$ and $\nabla_T (JV) = J(\nabla_T V) = 0$, we have

$$\nabla_T T = (-\phi' \sin\phi)\, V + (\phi' \cos\phi)\, JV = \phi'\, J\big(\cos\phi\, V + \sin\phi\, JV\big) = \phi'\, JT.$$

Since $\|T\| \equiv 1$, $\nabla_T T$ is orthogonal to T, hence it must be a multiple of JT; taking the JT–component gives

$$\kappa_g = \left\langle \nabla_T T,\ JT \right\rangle = \phi'.$$

$\square$

Corollary 11. *The total geodesic curvature of a curve on a surface is equal to the total rotation of a parallel vector field along the curve, i.e.*

$$\int_a^b \kappa_g = \phi(b) - \phi(a),$$

where ϕ is an angle function between T and V.

Exercise 12. Consider the circle of constant latitude ϕ on $\mathbf{S}^2$, which is given by

$$\alpha_\phi(\theta) := (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)),$$

where ϕ measures the angle with $(0,0,1)$. Show that the geodesic curvature of α_ϕ is $\cot(\phi)$. Conclude that if a vector is parallel translated along α, then its total rotation with respect to the tangent vector of α is $2\pi\cos(\phi)$.

Exercise 13. There is another way to compute the total rotation angle of a vector parallel translated along the circle of constant latitude α mentioned in Exercise 12. Consider a cone which is tangent to $\mathbf{S}^2$ along α. Cut this cone along a straight line through its apex and unroll it in the plane. Then α becomes a circular arc in the plane. Compute the total curvature of this arc. Why do you get the same answer as in Exercise 12?

Exercise 14. The French physicist Foucault observed that the plane in which a pendulum swings is not stationary by rotates with constant speed. Explain how one can determine the latitute of a city on earth from the total rotation of the plane of a pendulum over a 24 hour period.