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Differential Geometry I
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Lecture Notes 15

Riemannian Geodesics

Here we show that every Riemannian manifold admits a unique connection, called the Riemanninan or Levi-Civita connection, which satisfies two properties: symmetry, and compatibility with the metric, as we describe below. This result is known as the fundamental theorem of Rimeannian gemetry. Further we will show that the geodesics which arise from a Riemannian connection are locally minimize distance.

0.1 The bracket

For any pair of vector fields X, $Y \in \mathcal{X}(M)$ we may define a new vector field $[X,Y] \in \mathcal{X}(M)$ as follows. First recall that T_pM is isomorphic to D_pM the space of derivations of the germ of functions of M at. Thus we may define $[X,Y]$ by desrcribing how it acts on functions at each point:

$$[X,Y]_p f := X_p(Yf) - Y_p(Xf).$$

One may check that this does indeed define a derivation, i.e., $[X,Y]_p(\lambda f + g) = \lambda [X,Y]_p f + [X,Y]_p g$, and $[X,Y]_p(fg) = ([X,Y]_p f)g(p) + f(p)([X,Y]_p g)$. Further note that if $e_i(p) ;= e_i$ denotes the standard basis vector field of $\mathbf{R}^n$ then $[e_i, e_j] = 0$ (since partial derivatives commute). On the other hand it is not difficult to construct examples of vector fields whose bracket does not vanish:

Example 0.1. Let X, Y be vector fields on $\mathbf{R}^2$ given by $X(x,y) = (1,0)$ and $Y(x,y) = (0,x)$. Then

$$[X,Y]f = X\Big(x\frac{\partial f}{\partial y}\Big) - Y\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial f}{\partial y} + x\frac{\partial^2 f}{\partial x \partial y} - x\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f}{\partial y}$$

Lemma 0.2. *Let $f\colon M \to N$ be a diffeomorphism, and X, $Y \in \mathcal{X}(M)$. Then*

$$df([X,Y]) = [dfX, dfY].$$

Proof. Recall that for any vectorfield Z on M and function g on N, we have

$$\big((dfZ)g\big)(f(p)) = (dfZ)_{f(p)} g = (df_p Z)g = Z_p(g \circ f).$$

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Thus if we let $\overline{Z} := df Z$, and $\overline{p} := f(p)$, then

$$\Big((\overline{Z}g)\circ f\Big)(p) = (\overline{Z}g)(\overline{p}) = \overline{Z}_{\overline{p}}g = Z_p(g\circ f) = \Big(Z(g\circ f)\Big)(p).$$

Using the last set of identities, we may now compute

$$\begin{aligned}\Big(\overline{[X,Y]}g\Big)(\overline{p}) &= [X,Y]_p(g\circ f)\\ &= X_p\Big(Y(g\circ f)\Big) - Y_p\Big(X(g\circ f)\Big)\\ &= X_p\Big((\overline{Y}g)\circ f\Big) - Y_p\Big((\overline{X}g)\circ f\Big)\\ &= \overline{X}_{\overline{p}}(\overline{Y}g) - \overline{Y}_{\overline{p}}(\overline{X}g)\\ &= \Big([\overline{X},\overline{Y}]g\Big)(\overline{p}).\end{aligned}$$

□

Corollary 0.3. *Let (U,ϕ) be a local chart of M and $E_i(p) := d\phi^{-1}_{\phi(p)}(e_i)$ be the associated coordinate vector fields on U. Then $[E_i, E_j] = 0$.* □

Exercise 0.4. Show that the bracket satisfies the following properties:

$$[X,Y] = -[Y,X] \quad \text{and} \quad [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

0.2 Riemannian Connections

Recall that the standard connection in $\mathbf{R}^n$ is defined as

$$\nabla_X Y := (X(Y^1), \ldots, X(Y^n)).$$

Furthermore, recall that in $\mathbf{R}^n$, for any function $f\colon \mathbf{R}^n \to \mathbf{R}$ and vector field X we have

$$Xf = \langle X, \operatorname{grad} f\rangle.$$

Using these identites, we compute that

$$\begin{aligned}Z\langle X,Y\rangle &= \sum \big\langle Z, \operatorname{grad}(X^iY^i)\big\rangle\\ &= \sum \big\langle Z, \operatorname{grad}(X^i)Y^i + X^i \operatorname{grad}(Y^i))\big\rangle\\ &= \sum \big\langle Z, \operatorname{grad} X^i\big\rangle Y^i + \sum \big\langle Z, \operatorname{grad} Y^i\big\rangle X^i\\ &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.\end{aligned}$$

Motivated by this observation we say that a connection on a Riemannian manifold (M,g) is compatible with the metric provided that

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

Further note that

$$\langle Y, \nabla_X \operatorname{grad} f\rangle - \langle X, \nabla_Y \operatorname{grad} f\rangle = \sum_{i=1}^{n} Y^i X^i D_i f - \sum_{i=1}^{n} X^i Y^i D_i f = 0.$$

This property, together with the compatibility of ∇ with the innerproduct which we estalished above, may be used to compute that

$$\begin{aligned}(\nabla_X Y - \nabla_Y X)f &= \big\langle \nabla_X Y, \operatorname{grad} f\rangle - \langle \nabla_Y X, \operatorname{grad} f\rangle + \langle Y, \nabla_X \operatorname{grad} f\big\rangle - \langle X, \nabla_Y \operatorname{grad} f\rangle \\ &= X\langle Y, \operatorname{grad} f\rangle - Y\langle X, \operatorname{grad} f\rangle \\ &= X(Yf) - Y(Xf) \\ &= [X, Y](f).\end{aligned}$$

Thus we say that a connection on a manifold is *symmetric* provided that

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Exercise 0.5. Show that a connection is symmetric if and only the correspding Christoffel symbold satisfy

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

in every local chart.

If a connection is compatible with the metric and is symmetric we say that it is *Riemannian.* The following result is known as the fundamental theorem of Riemannian Geometry

Theorem 0.6. *Every Riemannian manifold admits a unique Riemannian connection.*

Proof. First suppose that the manifold (M, g) does admit some Riemannian connection ∇. We will show then that ∇ is unique. To see this, first note that, for any vector fields X, Y, $Z \in \mathcal{M}$,

$$\begin{aligned}Zg(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y), \\ Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\ Yg(Z,X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X).\end{aligned}$$

This yields that

$$\begin{aligned}&Zg(X,Y) + Xg(Y,Z) - Yg(Z,X) \\ &\qquad = g([X,Z],Y) + g([Y,Z],X) + g([X,Y],Z) + 2g(Z, \nabla_Y X).\end{aligned}$$

Therefore

$$g(Z, \nabla_Y X) = \frac{1}{2}\Big(Zg(X,Y) + Xg(Y,Z) - Yg(Z,X) \\ - g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)\Big).$$

This shows that $\nabla_Y X$ is completely determined by g, so it must be unique.

To prove existence, now note that we may define ∇ by using the last expression displayed above. It is easy to check that ∇ would then be a Riemannian connection. □

Next we are going to derive the local expression for the Christophel symbols associated to a Riemannian connection. Let (U, ϕ) be a local chart of M and $E_i(p) := d\phi^{-1}_{\phi(p)}(e_i)$ be the corresponding coordinate vector fields on U. Then, recalling the $[E_i, E_j] = 0$, the last displayed expression yields that

$$g\Big(E_k, \sum_\ell \Gamma^\ell_{ij} E_\ell\Big) = \frac{1}{2}\Big(E_k g(E_i, E_j) + E_i g(E_j, E_k) - E_j g(E_k, E_i)\Big).$$

Now set $g_{ij} := g(E_i, E_j)$. Further recall that if $f \colon M \to \mathbf{R}$ is any function then $E_i f(p) = D_i(f \circ \phi^{-1})(\phi(p))$. Thus if we set $\overline{f} := f \circ \phi^{-1}$, then $E_i(f)$, then we have $E_i f(p) = D_i \overline{f}(\phi(p))$, and the last expression may be rewritten as:

$$\sum_\ell \overline{\Gamma}^\ell_{ij} \overline{g}_{k\ell} = \frac{1}{2}\Big(D_k \overline{g}_{ij} + D_i \overline{g}_{jk} - D_j \overline{g}_{ki}\Big).$$

Now let g^{ij} be the coefficients of the matrix which is the inverse of the matrix with coefficients g_{ij}. Then $\sum_k \overline{g}_{k\ell} g^{km} = \delta_{\ell m}$ where $\delta_{\ell m}$ are the coefficients of the identity matrix. Therefore

$$\sum_{k\ell} \overline{\Gamma}^\ell_{ij} \overline{g}_{k\ell} \overline{g}^{km} = \sum_\ell \overline{\Gamma}^\ell_{ij} \delta_{\ell m} = \overline{\Gamma}^m_{ij}.$$

This yields that

$$\overline{\Gamma}^m_{ij} = \frac{1}{2} \sum_k \overline{g}^{km}\Big(D_k \overline{g}_{ij} + D_i \overline{g}_{jk} - D_j \overline{g}_{ki}\Big). \tag{1}$$

0.3 Induced connection on Riemanninan submanifolds

Recall that if $\overline{M}$ is a manifold with connection $\overline{\nabla}$, then any submanifold $M \subset \overline{M}$ inherits a connection ∇ given by

$$\nabla_X Y := \big(\overline{\nabla}_{\overline{X}} \overline{Y}\big)^\top.$$

Further recall that, if $(\overline{M}, \overline{g})$ is a Riemannian manifold, then M inherits a Riemannian metric g given by

$$g(X, Y) := \overline{g}(\overline{X}, \overline{Y}).$$

Thus one may ask that if $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$, then is ∇ a Riemannian connection, i.e., is it symmetric and is compatible with g? Here we show that the answer is yes:

Proposition 0.7. *The induced connection on a Riemannian submanifold is Riemannian.*

Proof. Let $p \in M$, X, Y be vector fields on M, and $\overline{X}$, $\overline{Y}$ be their extensions to a neighborhood $U \subset \overline{M}$ of p. Then

$$\begin{aligned} Z_p g(X,Y) &= Z_p \overline{g}(\overline{X}, \overline{Y}) \\ &= \overline{g}(\overline{\nabla}_{Z_p}\overline{X}, Y_p) + \overline{g}(X_p, \overline{\nabla}_{Z_p}\overline{Y}) \\ &= \overline{g}((\overline{\nabla}_{Z_p}\overline{X})^\top, Y_p) + \overline{g}(X_p, (\overline{\nabla}_{Z_p}\overline{Y})^\top) \\ &= g(\nabla_{Z_p}X, Y_p) + g(X_p, \nabla_{Z_p}Y) \end{aligned}$$

So ∇ is compatible with g. Next note that

$$\nabla_{X_p}Y - \nabla_{Y_p}X = (\nabla_{\overline{X}_p}\overline{Y})^\top - (\nabla_{\overline{Y}_p}\overline{X})^\top = [\overline{X}, \overline{Y}]_p^\top.$$

But if $\overline{f}$ is any function on $\overline{M}$ and f is its restriction to M, then

$$[\overline{X}, \overline{Y}]_p \overline{f} = X_p(\overline{Y f}) - Y_p(\overline{X f}) = X_p(Yf) - Y_p(Xf) = [X,Y]_p f.$$

Thus
$$[\overline{X}, \overline{Y}]_p^\top = [X,Y]_p^\top = [X,Y]_p.$$
So ∇ is symmetric. □

0.4 Speed of Geodesics

If (M,g) is a Riemannian metric, we say that a curve $c\colon I \to M$ is a (Riemannian) geodesic provided that g is a geodesic with respect to the Riemannian connection of M.

Lemma 0.8. *Every Riemannian geodesic $c\colon I \to M$ has constant speed, i.e., $g(c'(t), c'(t))$ is constant.*

Proof. Let $\overline{c}'$ be a vector field in a neighborhood U of $c(t_0)$ such that $\overline{c}'(c(t)) = c'(t)$, for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$. Now define $f\colon U \to \mathbf{R}$ by $f(p) = g(\overline{c}'(p), \overline{c}'(p))$. Then $g(c'(t), c'(t)) = f(c(t))$, and it follows that

$$\frac{d}{dt} g(c'(t), c'(t))\Big|_{t=t_0} = (f \circ c)'(t_0) = c'(t_0)\big[g(\overline{c}', \overline{c}')\big] = 2\, g\big(\nabla_{c'(t_0)}\overline{c}', \overline{c}'(t_0)\big) = 0.$$

Thus $g(c'(t), c'(t))$ is constant. □

If $c\colon I \to M$ is any curve, then we say that $\overline{c}\colon J \to M$ is a *reparametrization* of c provided that $\overline{c} = c \circ u$ for some diffeomorphism $u\colon J \to I$.

Lemma 0.9. *If $c\colon I \to M$ is a geodesic, then so is any reparametrization $\overline{c} = c \circ u$, where $u(t) = kt + t_0$ for some constants k and t_0.*

Proof. The chain rule yields that

$$\overline{c}'(t) = d\overline{c}_t(1) = dc_{u(t)} \circ du_t(1) = dc_{u(t)}(u'(t)) = dc_{u(t)}(k) = kdc_{u(t)}(1) = kc'(u(t)).$$

Consequently,

$$\nabla_{\overline{c}'(t)}\overline{c}' = \nabla_{kc'(kt+t_0)}kc' = k^2\nabla_{c'(kt+t_0)}c' = 0.$$

□

Proposition 0.10. *Let $c\colon I \to M$ be a geodesic. Then any reparamterization $\overline{c}\colon J \to M$ of c is a geodesic as well, if and only if it has constant speed.*

Proof. If $\overline{c}$ is a geodesic, then it must have constant speed as we showed earlier. Now suppose that $\overline{c}$ has constant speed. Further note that, since $\overline{c} = c \circ u$, for some diffeomorphism $u\colon J \to I$, it follows that

$$\overline{c}'(t) = d\overline{c}_t(1) = dc_{u(t)} \circ du_t(1) = dc_{u(t)}(u'(t)) = u'(t)dc_{u(t)}(1) = u'(t)c'(u(t)).$$

Thus, since $\overline{c}'$ and c' both have constant magnitudes, it follows that u' is constant. But then $u(t) = kt + t_0$, and the previous lemma implies that $\overline{c}$ is a geodesic. □

0.5 Example: Geodesics of $\mathbf{H}^2$

Here we show that the (nontrivial) geodesics in the Poincare's upper half-plane either trace vertical lines or semicircles which meet the x-axis orthogonally. To this end, we first recall that the standard (hyperbolic) metric on the upper half plane is given by

$$g_{(x,y)}(X, Y) = \frac{\langle X, Y\rangle}{y^2}.$$

Thus

$$g_{11}(x, y) = \frac{1}{y^2}, \qquad g_{12}(x, y) = g_{21}(x, y) = 0, \qquad g_{22}(x, y) = \frac{1}{y^2}.$$

Further

$$g^{11}(x, y) = y^2, \qquad g^{12}(x, y) = g^{21}(x, y) = 0, \qquad g^{22}(x, y) = y^2.$$

Now note that we may let the local chart ϕ to be the identity function. Then $\overline{\Gamma}_{ij}^m = \Gamma_{ij}^m$, and so using (1) we may compute that

$$\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 =, 0 \qquad \Gamma_{11}^2 = \frac{1}{y}, \qquad \Gamma_{12}^1 = \Gamma_{22}^2 = -\frac{1}{y}.$$

Now recall that $c\colon I \to \mathbf{H}^2$ is a geodesic if the following equations are satisfied:

$$\ddot{c}^k(t) + \sum_{ij} \dot{c}^i(t)\dot{c}^j(t)\Gamma_{ij}^k(c(t)) = 0.$$

So if $c(t) = (x(t), y(t))$, then we have

$$\ddot{x} - 2\frac{\dot{x}\dot{y}}{y} = 0, \qquad \ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} = 0. \tag{2}$$

To find the solution to these equations, subject to initial conditions $c(0) = (x_0, y_0)$ and $\dot{c}(0) = (\dot{x}_0, \dot{y}_0)$, first suppose that $\dot{x}_0 = 0$. Then the second equation reduces to $\dot{y}/y = const$. Thus, when $\dot{x}_0 = 0$, then either c traces a vertical line (if $\dot{y}_0 \neq 0$) or is just a point (if $\dot{y}_0 = 0$). It remains then to consider the case when $\dot{x}_0 \neq 0$. We claim that in this case c traces a part of a circle centered at a point on the x-axis, i.e.,

$$(x - a)^2 + y^2 = const$$

for some constant a (in particular, when $\dot{x}_0 \neq 0$, then $\dot{y}_0 \neq 0$ as well, which may be readily seen from the second equation in (2)). Differentiating both sides of the above equality yields that the above equality holds if and only if

$$a = x + \frac{y\dot{y}}{\dot{x}}.$$

So we just need to check that a is indeed constant, which is a matter of a simple computation with the aid of (2):

$$\dot{a} = \dot{x} + \frac{(\dot{y}^2 + y\ddot{y})\dot{x} - y\dot{y}\ddot{x}}{\dot{x}^2} = \dot{x} + \frac{(\dot{y}^2 + \dot{y}^2 - \dot{x}^2)\dot{x} - \dot{y}(2\dot{x}\dot{y})}{\dot{x}^2} = 0.$$