Math 497C
Curves and Surfaces
Fall 2004, PSU

Mar 3, 2004[1]

Lecture Notes 11

2.4 Intrinsic Metric and Isometries of Surfaces

Let $M \subset \mathbf{R}^3$ be a regular embedded surface and p, $q \in M$, then we define

$$\mathrm{dist}_M(p,q) := \inf\{\mathrm{Length}[\gamma] \mid \gamma\colon [0,1] \to M, \gamma(0) = p, \gamma(1) = q\}.$$

Exercise 1. Show that (M, dist_M) is a metric space.

Lemma 2. *Show that if M is a C^1 surface, and $X \subset M$ is compact, then for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\Big|\,\mathrm{dist}_M(p,q) - \|p-q\|\,\Big| \leq \epsilon\|p-q\|$$

for all p, $q \in X$, with $\mathrm{dist}_M(p,q) \leq \delta$.

Proof. Define $F\colon M \times M \to \mathbf{R}$ by $F(p,q) := \mathrm{dist}_M(p,q)/\|p-q\|$, if $p \neq q$ and $F(p,q) := 1$ otherwise. We claim that F is continuous. To see this let p_i be a sequence of points of M which converge to a point $p \in M$. We may assume that p_i are contained in a Monge patch of M centered at p given by

$$X(u_1,u_2) = (u_1, u_2, h(u_1,u_2)).$$

Let x_i and y_i be the x and y coordinates of p_i. If p_i is sufficiently close to $p = (0,0)$, then, since $\nabla h(0,0) = 0$, we can make sure that

$$\|\nabla h(tx_i, ty_i)\|^2 \leq \epsilon,$$

for all $t \in [0,1]$ and $\epsilon > 0$. Let $\gamma\colon [0,1] \to \mathbf{R}^3$ be the curve given by

$$\gamma(t) = (tx_i, ty_i, h(tx_i, ty_i)).$$

[1]Last revised: October 27, 2021

Then, since γ is a curve on M,

$$\begin{aligned}
\mathrm{dist}_M(p, p_i) &\le \mathrm{Length}[\gamma] \\
&= \int_0^1 \sqrt{x_i^2 + y_i^2 + \langle \nabla h(tx_i, ty_i), (x_i, y_i)\rangle^2}\, dt \\
&\le \int_0^1 \sqrt{x_i^2 + y_i^2 + \epsilon (x_i^2 + y_i^2)^2}\, dt \\
&\le \sqrt{1+\epsilon}\sqrt{x_i^2 + y_i^2} \\
&\le (1+\epsilon)\|p - p_i\|
\end{aligned}$$

So, for any $\epsilon > 0$ we have

$$1 \le \frac{\mathrm{dist}_M(p, p_i)}{\|p - p_i\|} \le 1 + \epsilon$$

provided that p_i is sufficiently close to p. We conclude then that F is continuous. So $U := F^{-1}([1, 1+\epsilon])$ is an open subset of $M \times M$ which contains the diagonal $\Delta_M := \{(p,p) \mid p \in M\}$. Since $\Delta_X \subset \Delta_M$ is compact, we may then choose δ so small that $V_\delta(\Delta_X) \subset U$, where $V_\delta(\Delta_X)$ denotes the open neighborhood of Δ_X in $M \times M$ which consists of all pairs of points (p, q) with $\mathrm{dist}_M(p, q) \le \delta$. $\square$

Exercise 3. Does the above lemma hold also for C^0 surfaces?

If $\gamma \colon [a, b] \to M$ is any curve then we may define

$$\mathrm{Length}_M[\gamma] := \sup \left\{ \sum_{i=1}^{k} \mathrm{dist}_M(\gamma(t_i), \gamma(t_{i-1})) \,\middle|\, \{t_0, \dots, t_k\} \in \mathrm{Partition}[a, b] \right\}.$$

Lemma 4. $\mathrm{Length}_M[\gamma] = \mathrm{Length}[\gamma]$.

Proof. Note that

$$\mathrm{dist}_M(\gamma(t_i), \gamma(t_{i-1})) \ge \|\gamma(t_i) - \gamma(t_{i-1})\|.$$

Thus $\mathrm{Length}_M[\gamma] \ge \mathrm{Length}[\gamma]$. Further, by the previous lemma, we can make sure that

$$\mathrm{dist}_M(\gamma(t_i), \gamma(t_{i-1})) \le (1+\epsilon)\|\gamma(t_i) - \gamma(t_{i-1})\|,$$

which yields $\mathrm{Length}_M[\gamma] \le (1+\epsilon)\,\mathrm{Length}[\gamma]$, for any $\epsilon > 0$. $\square$

We say that $f\colon M \to \overline{M}$ is an *isometry* provided that

$$\mathrm{dist}_{\overline{M}}(f(p), f(q)) = \mathrm{dist}_M(p, q).$$

Lemma 5. *$f\colon M \to \overline{M}$ is an isometry, if and only if* $\mathrm{Length}[\gamma] = \mathrm{Length}[f \circ \gamma]$ *for all curves $\gamma\colon [a, b] \to M$.*

Proof. If f is an isometry, then, by the previous lemma,

$$\mathrm{Length}[\gamma] = \mathrm{Length}_M[\gamma] = \mathrm{Length}_{\overline{M}}[f \circ \gamma] = \mathrm{Length}_M[f \circ \gamma].$$

The converse is clear. □

Exercise 6. Justify the middle equality in the last expression displayed above.

Theorem 7. *$f\colon M \to \overline{M}$ is an* isometry *if and only if for all $p \in M$, and $v, w \in T_pM$,*

$$\big\langle df_p(v), df_p(w)\big\rangle = \langle v, w\rangle.$$

Proof. Suppose that f is an isometry. Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$, and $\gamma'(0) = v$. Then, by the previous lemma

$$\int_{-\epsilon}^{\epsilon} \|\gamma'(t)\|\, dt = \int_{-\epsilon}^{\epsilon} \|(f \circ \gamma)'(t)\|\, dt$$

Taking the limit of both sides as $\epsilon \to 0$ and applying the mean value theorem for integrals, yields then that

$$\|v\| = \|\gamma'(0)\| = \|(f \circ \gamma)'(0)\| = \|df_p(v)\|.$$

Thus df preserves the norm, which implies that it must preserve the inner-product as well (see the following exercise).

Conversely, suppose that $\|v\| = \|df_p(v)\|$. Then, if $\gamma\colon [a, b] \to M$ is any curve, we have

$$\int_a^b \|(f \circ \gamma)'(t)\| dt = \int_a^b \|df_{\gamma(t)}(\gamma'(t))\| dt = \int_a^b \|\gamma'(t)\| dt.$$

So f preserves the length of all curves, which, by the previous Lemma, shows that f is an isometry. □

Exercise 8. Show that a linear function $F\colon \mathbf{R}^n \to \mathbf{R}^n$ preserves the norm $\|\cdot\|$ if and only if it preserves the inner product $\langle\cdot, \cdot\rangle$.

2.5 Gauss's Theorem Egregium

Lemma 9. *Let $X : U \to M$ be a proper regular chart. Then $\overline{X} := f \circ X : U \to \overline{M}$ is a proper regular chart as well and $g_{ij} = \overline{g}_{ij}$ on U.*

Proof. Using the last theorem we have

$$\begin{aligned}
\overline{g}_{ij}(u_1, u_2) &= \langle D_i\overline{X}(u_1, u_2), D_j\overline{X}(u_1, u_2)\rangle \\
&= \langle D_i(f \circ X)(u_1, u_2), D_j(f \circ X)(u_1, u_2)\rangle \\
&= \langle df_{X(u_1,u_2)}(D_iX(u_1, u_2)), df_{X(u_1,u_2)}(D_jX(u_1, u_2))\rangle \\
&= \langle D_iX(u_1, u_2), D_jX(u_1, u_2)\rangle \\
&= g_{ij}(u_1, u_2).
\end{aligned}$$

□

Exercise 10. Justify the third equality in the last displayed expression above.

Let $\mathcal{F}$ denote the set of functions $f : U \to \mathbf{R}$ where $U \subset \mathbf{R}^2$ is an open neighborhood of the origin.

Lemma 11. *There exists a mapping* Briochi: $\mathcal{F} \times \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ *such that for any chart $X : U \to M$ centered at $p \in M$,*

$$K(p) = \text{Briochi}[g_{11}, g_{12}, g_{22}](0, 0).$$

Proof. Recall that

$$K(p) = \frac{\det l_{ij}(0, 0)}{\det g_{ij}(0, 0)},$$

and, by Lagrange's identity,

$$l_{ij} = \left\langle X_{ij}, \frac{X_1 \times X_2}{\|X_1 \times X_2\|} \right\rangle = \frac{1}{\sqrt{\det g_{ij}}} \langle X_{ij}, X_1 \times X_2\rangle\,,$$

where $X_{ij} := D_{ij}X$, and $X_i := D_iX$. Thus

$$K(p) = \frac{\det(\langle X_{ij}(0, 0), X_1(0, 0) \times X_2(0, 0)\rangle)}{(\det g_{ij}(0, 0))^2}.$$

Next note that

$$\det(\langle X_{ij}, X_1 \times X_2\rangle) = \langle X_{11}, X_1 \times X_2\rangle\langle X_{22}, X_1 \times X_2\rangle - \langle X_{12}, X_1 \times X_2\rangle^2$$

The right hand side of the last expression may be rewritten as

$$\det(X_{11}, X_1, X_2)\det(X_{22}, X_1, X_2) - (\det(X_{12}, X_1, X_2))^2,$$

where (u, v, w) here denotes the matrix with columns u, v, and w. Recall that if A is a square matrix with transpose A^T, then $\det A = \det A^T$. Thus the last expression displayed above is equivalent to

$$\det((X_{11}, X_1, X_2)^T(X_{22}, X_1, X_2)) - \det((X_{12}, X_1, X_2)^T(X_{12}, X_1, X_2)),$$

which in turn can be written as

$$\det\begin{pmatrix} \langle X_{11}, X_{22}\rangle & \langle X_{11}, X_1\rangle & \langle X_{11}, X_2\rangle \\ \langle X_1, X_{22}\rangle & \langle X_1, X_1\rangle & \langle X_1, X_2\rangle \\ \langle X_2, X_{22}\rangle & \langle X_2, X_1\rangle & \langle X_2, X_2\rangle \end{pmatrix} - \det\begin{pmatrix} \langle X_{12}, X_{12}\rangle & \langle X_{12}, X_1\rangle & \langle X_{12}, X_2\rangle \\ \langle X_1, X_{12}\rangle & \langle X_1, X_1\rangle & \langle X_1, X_2\rangle \\ \langle X_2, X_{12}\rangle & \langle X_2, X_1\rangle & \langle X_2, X_2\rangle \end{pmatrix}.$$

If we expand the above determinants along their first rows, then $\langle X_{11}, X_{22}\rangle$ and $\langle X_{12}, X_{22}\rangle$ will have the same coefficients. This implies that we can rewrite the last expression as

$$\det\begin{pmatrix} \langle X_{11}, X_{22}\rangle - \langle X_{12}, X_{12}\rangle & \langle X_{11}, X_1\rangle & \langle X_{11}, X_2\rangle \\ \langle X_1, X_{22}\rangle & \langle X_1, X_1\rangle & \langle X_1, X_2\rangle \\ \langle X_2, X_{22}\rangle & \langle X_2, X_1\rangle & \langle X_2, X_2\rangle \end{pmatrix} - \det\begin{pmatrix} 0 & \langle X_{12}, X_1\rangle & \langle X_{12}, X_2\rangle \\ \langle X_1, X_{12}\rangle & \langle X_1, X_1\rangle & \langle X_1, X_2\rangle \\ \langle X_2, X_{12}\rangle & \langle X_2, X_1\rangle & \langle X_2, X_2\rangle \end{pmatrix}.$$

Now note that each of the entries in the above matrices can be expressed purely in terms of g_{ij}, since

$$\langle X_{ii}, X_j\rangle = \langle X_i, X_j\rangle_i - \langle X_i, X_{ji}\rangle = (g_{ij})_i - \frac{1}{2}(g_{ii})_j,$$
$$\langle X_{ij}, X_i\rangle = \frac{1}{2}\langle X_i, X_i\rangle_j = \frac{1}{2}(g_{ii})_j,$$

and

$$\begin{aligned} \langle X_{11}, X_{22}\rangle - \langle X_{12}, X_{12}\rangle &= \langle X_1, X_{22}\rangle_1 - \langle X_1, X_{12}\rangle_2 \\ &= (g_{21})_{21} - \frac{1}{2}(g_{11})_{21} - \frac{1}{2}(g_{11})_2. \end{aligned}$$

Substituting the above values in the previous matrices, we define

$$\begin{aligned}
&\mathrm{Briochi}[g_{11}, g_{22}, g_{33}] := \\
&\frac{1}{(\det(g_{ij}))^2}\Bigg(\det \begin{pmatrix} (g_{21})_{21} - \frac{1}{2}(g_{11})_{21} - \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{11})_1 & \frac{1}{2}(g_{11})_2 \\ (g_{21})_2 - \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_2 & g_{21} & g_{22} \end{pmatrix} \\
&\qquad - \det \begin{pmatrix} 0 & \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{22})_1 \\ \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_1 & g_{21} & g_{22} \end{pmatrix} \Bigg).
\end{aligned}$$

Evaluating the above expression at $(0,0)$ yields that Gaussian curvature $K(p)$. □

Theorem 12. *If $f\colon M \to \overline{M}$ is an isometry, then $\overline{K}(f(p)) = K(p)$, where K and $\overline{K}$ denote the Gaussian curvatures of M and $\overline{M}$ respectively.*

Proof. Let $X\colon U \to M$ be a chart centered at p, then $\overline{X} := f \circ X$ is a chart of $\overline{M}$ centered at $f(p)$. Let g_{ij} and $\overline{g}_{ij}$ denote the coefficients of the first fundamental form with respect to the charts X and $\overline{X}$ respectively. Then, using the previous two lemmas, we have

$$\begin{aligned}
\overline{K}(f(p)) &= \mathrm{Briochi}[\overline{g}_{11}, \overline{g}_{12}, \overline{g}_{22}](0,0) \\
&= \mathrm{Briochi}[g_{11}, g_{12}, g_{22}](0,0) \\
&= K(p).
\end{aligned}$$

□

Exercise 13. Let $M \subset \mathbf{R}^3$ be a regular embedded surface and $p \in M$. Suppose that $K(p) \neq 0$. Does there exist a chart $X\colon U \to M$ such that D_1X and D_2X are orthonormal at all points of U.