Math 598 Mar 20, 2005[1]
Geometry and Topology II
Spring 2005, PSU

Lecture Notes 10

3.4 Proof of Sard's Theorem

This section will be typeset later. In the meantime the reader is refered to Milnor's book on *Topology from Differentiable View Point*.

Exercise 1. Show that Sard theorem immediately implies that if $f\colon M \to N$ is a smooth function and $\dim(M) < \dim(N)$ then $f(M)$ has measure zero.

Exercise 2. Use Sard's theorem to show that $\mathbf{S}^n$ is simply connected for $n \geq 2$. (Hint: it is enough to show that every continuos map $f\colon \mathbf{S}^1 \to \mathbf{S}^2$ is homotopic to a map $\overline{f}\colon \mathbf{S}^1 \to \mathbf{S}^2$ which is not onto. You also need to use Wierstrauss's approximation theorem.)

Exercise 3. Let M^n be a compact manifold smoothly embedded in $\mathbf{R}^{n+1}$. Show that almost every hyperplane $H \subset \mathbf{R}^{n+1}$ is transversal to M, i.e., H is not tangent to M at any points (Hint: consider the unit normal vector field $\nu\colon M \to \mathbf{S}^n$.)

Exercise 4. Show that if $X \subset \mathbf{R}^n$ is a measurable set such that the intersection of X with any horizontal hyperplane $(constant) \times \mathbf{R}^{n-1}$ has measure zero, then X has measure zero (this was one of the facts used in the proof of Sard's theorem).

Exercise 5. Show that to prove Sard's theorem it suffices to consider the case of mappings $f\colon U \to \mathbf{R}^m$, where $U \subset \mathbf{R}^n$.

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