Math 734 — Aug 22, 2002[1]
Differential Geometry
Fall 2002, USC

Lecture Notes 1

1 Topological Manifolds

The basic objects of study in this class are manifolds. Roughly speaking, these are objects which locally resemble a Euclidean space. In this section we develop the formal definition of manifolds and construct many examples.

1.1 The Euclidean space

By $\mathbf{R}$ we shall always mean the set of real numbers. The set of all n-tuples of real numbers $\mathbf{R}^n := \{(p^1, \dots, p^n) \mid p^i \in \mathbf{R}\}$ is called the Euclidean n-space. So we have

$$p \in \mathbf{R}^n \iff p = (p^1, \dots, p^n), \quad p^i \in \mathbf{R}.$$

Let p and q be a pair of points (or vectors) in $\mathbf{R}^n$. We define $p + q := (p^1 + q^1, \dots, p^n + q^n)$. Further, for any scalar $r \in \mathbf{R}$, we define $rp := (rp^1, \dots, rp^n)$. It is easy to show that the operations of addition and scalar multiplication that we have defined turn $\mathbf{R}^n$ into a vector space over the field of real numbers. Next we define the standard inner product on $\mathbf{R}^n$ by

$$\langle p, q \rangle = p^1 q^1 + \ldots + p^n q^n.$$

Note that the mapping $\langle \cdot, \cdot \rangle \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is linear in each variable and is symmetric. The standard inner product induces a norm on $\mathbf{R}^n$ defined by

$$\|p\| := \langle p, p \rangle^{1/2}.$$

If $p \in \mathbf{R}$, we usually write $|p|$ instead of $\|p\|$.

Exercise 1.1.1. (The Cauchy-Schwartz inequality) Prove that $|\langle p, q \rangle| \leqslant \|p\| \, \|q\|$, for all p and q in $\mathbf{R}^n$ (*Hints:* If p and q are linearly dependent the solution is clear. Otherwise, let $f(\lambda) := \langle p - \lambda q, p - \lambda q \rangle$. Then $f(\lambda) > 0$. Further, note that $f(\lambda)$ may be written as a quadratic equation in λ. Hence its discriminant must be negative).

The standard Euclidean distance in $\mathbf{R}^n$ is given by

$$\operatorname{dist}(p, q) := \|p - q\|.$$

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Exercise 1.1.2. (The triangle inequality) Show that $\mathrm{dist}(p,q) + \mathrm{dist}(q,r) \geqslant \mathrm{dist}(p,r)$ for all p, q in $\mathbf{R}^n$. (*Hint:* use the Cauchy-Schwartz inequality).

By a *metric* on a set X we mean a mapping $d\colon X \times X \to \mathbf{R}$ such that

1. $d(p,q) \geqslant 0$, with equality if and only if $p = q$.

2. $d(p,q) = d(q,p)$.

3. $d(p,q) + d(q,r) \geqslant d(p,r)$.

These properties are called, respectively, positive-definiteness, symmetry, and the triangle inequality. The pair (X,d) is called a *metric space*. Using the above exercise, one immediately checks that $(\mathbf{R}^n, \mathrm{dist})$ is a metric space. Geometry, in its broadest definition, is the study of metric spaces.

Finally, we define the *angle* between a pair of vectors in $\mathbf{R}^n$ by

$$\mathrm{angle}(p,q) := \cos^{-1}\frac{\langle p,q\rangle}{\|p\|\,\|q\|}.$$

Note that the above is well defined by the Cauchy-Schwartz inequality.

Exercise 1.1.3. (The Pythagorean theorem) Show that in a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the length of the sides (*Hint:* First prove that whenever $\langle p,q\rangle = 0$, $\|p\|^2 + \|q\|^2 = \|p-q\|^2$. Then show that this proves the theorem.).

Exercise 1.1.4. Show that the sum of angles in a triangle is π.

1.2 Topological spaces

By a *topological space* we mean a set X together with a collection T of subsets of X which satisfy the following properties:

1. $X \in T$, and $\emptyset \in T$.

2. If U_1, $U_2 \in T$, then $U_1 \cap U_2 \in T$.

3. If $U_i \in T$, $i \in I$, then $\cup_i U \in T$.

The elements of T are called *open* sets. Note that property 2 implies that any <u>finite</u> intersection of open sets is open, and property 3 states that the union of any collection of open sets is open. Any collection of subsets of X satisfying the above properties is called a *topology* on X.

Exercise 1.2.1 (Metric Topology). Let (X, d) be a metric space. For any $p \in X$, and $r > 0$ define the ball of radius r centered at p as

$$B_r(p) := \{x \in X \mid d(x,p) \leq r\}.$$

We say $U \subset X$ is open if for each point p of U there is an $r > 0$ such that $B_r(p) \subset U$. Show that this defines a topology on X. In particular, $(\mathbf{R}^n, \mathrm{dist})$ is a topological space.

Thus every metric space is a topological space. The converse, however, is not true. See Appendix A in Spivak.

Exercise 1.2.2. Show that the intersection of an infinite collection of open subsets of $\mathbf{R}^n$ may not be open.

Let o denote the *origin* of $\mathbf{R}^n$, that is

$$o := (0, \dots, 0).$$

The n-dimensional Euclidean sphere is defined as

$$\mathbf{S}^n := \{\, x \in \mathbf{R}^{n+1} \mid \mathrm{dist}(x, o) = 1 \,\}.$$

The next exercise shows how we may define a topology on $\mathbf{S}^n$.

Exercise 1.2.3 (Subspace Topology). Let X be a topological space and suppose $Y \subset X$. Then we say that a subset V of Y is open if there exists an open subset U of X such that $V = U \cap Y$. Show that with this collection of open sets, Y is a topological space.

The n-dimensional torus T^n is defined as the cartesian product of n copies of $\mathbf{S}^1$,

$$T^n := \mathbf{S}^1 \times \cdots \times \mathbf{S}^1.$$

The next exercise shows that T^n admits a natural topology:

Exercise 1.2.4 (The Product Topology). Let X_1 and X_2 be topological spaces, and $X_1 \times X_2$ be their Cartesian product, that is

$$X_1 \times X_2 := \{\, (x_1, x_2) \mid x_1 \in X_1 \text{ and } x_2 \in X_2 \,\}.$$

We say that $U \subset X_1 \times X_2$ is open if $U = U_1 \times U_2$ where U_1 and U_2 are open subsets of X_1 and X_2 respectively. Show that this defines a topology on $X_1 \times X_2$.

A *partition* P of a set X is defined as a collection P_i, $i \in I$, of subsets of X such that $X \subset \cup_i P_i$ and $P_i \cap P_j = \emptyset$ whenever $i \neq j$. For any $x \in X$, the element of P which contains x is called the equivalence class of x and is denoted by $[x]$. Thus we get a mapping $\pi \colon X \to P$ given by $\pi(x) := [x]$. Suppose that X is a topological space. Then we say that a subset U of P is open if $\pi^{-1}(U)$ is open in X.

Exercise 1.2.5 (Quotient Topology). Let X be a topological space and P be a partition of X. Show that P with the collection of open sets defined above is a topological space.

Exercise 1.2.6 (Torus). Let P be a partition of $[0,1] \times [0,1]$ consisting of the following sets: (i) all sets of the form $\{(x,y)\}$ where $(x,y) \in (0,1)\times(0,1)$; (ii) all sets of the form $\{(x,1),(x,0)\}$ where $x \in (0,1)$; (iii) all sets of the form $\{(1,y),(0,y)\}$ where $y \in (0,1)$; and (iv) the set $\{(0,0),(0,1),(1,0),(1,1)\}$. Sketch the various kinds of open sets in P under its quotient topology.

1.3 Homeomorphisms

A mapping $f\colon X \to Y$ between topological spaces is *continuous* if for every open set $U \subset X$, $f^{-1}(U)$ is open in Y. Intuitively, we may think of a continuous map as one which sends nearby points to nearby points.

Exercise 1.3.1. Let A, $B \subset \mathbf{R}^n$ be arbitrary subsets, $f\colon A \to B$ be a continuous map, and $p \in A$. Show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\mathrm{dist}(x,p) < \delta$, then $\mathrm{dist}(f(x), f(p)) < \epsilon$.

We say that two topological spaces X and Y are *homeomorphic* if there exists a bijection $f\colon X \to Y$ which is continuous and has a continuous inverse. The main problem in topology is deciding when two topological spaces are homeomorphic.

Exercise 1.3.2. Show that $\mathbf{S}^n - \{(0,0,\dots,1)\}$ is homeomorphic to $\mathbf{R}^n$.

Exercise 1.3.3. Let $X := [1,0] \times [1,0]$, T_1 be the subspace topology on X induced by $\mathbf{R}^2$ (see Exercise 1.2.3), T_2 be the product topology (see Exercise 1.2.4), and T_3 be the quotient topology of Exercise 1.2.6. Show that (X,T_1) is homeomorphic to (X,T_2), but (X,T_3) is not homeomorphic to either of these spaces.

The n-dimensional Euclidean *open ball* of radius r centered at p is defined by

$$U_r^n(p) := \{x \in \mathbf{R}^n \mid \mathrm{dist}(x,p) < r\}.$$

Exercise 1.3.4. Show that $U_1^n(o)$ is homeomorphic to $\mathbf{R}^n$.

For any a, $b \in \mathbf{R}$, we set

$$[a,b] := \{x \in \mathbf{R} \mid a \leq x \leq b\},$$

and

$$(a,b) := \{x \in \mathbf{R} \mid a < x < b\}.$$

Exercise 1.3.5. Let P be a partition of $[0,1]$ consisting of all sets $\{x\}$ where $x \in (0,1)$ and the set $\{0,1\}$. Show that P, with respect to its quotient topology, is homeomorphic to $\mathbf{S}^1$ (Hint: consider the mapping $f\colon [0,1] \to \mathbf{S}^1$ given by $f(t) = e^{2\pi i t}$).

Exercise 1.3.6. Let P be the partition of $[0,1] \times [0,1]$ described in Exercise 1.2.6. Show that P, with its quotient topology, is homeomorphic to T^2.

Let P be the partition of $\mathbf{S}^n$ consisting of all sets of the form $\{p, -p\}$ where $p \in \mathbf{S}^n$. Then P with its quotient topology is called the real projective space of dimension n and is denoted by $\mathbf{RP}^n$.

Exercise 1.3.7. Let P be a partition of $B_1^2(o)$ consisting of all sets $\{x\}$ where $x \in U_1^2(o)$, and the all the sets $\{x, -x\}$ where $x \in \mathbf{S}^1$. Show that P, with its quotient topology, is homeomorphic to $\mathbf{RP}^2$.

Next we show that $\mathbf{S}^n$ is not homeomorphic to $\mathbf{R}^m$. This requires us to recall the notion of compactness.

We say that a collection of subsets of X *cover* X, if X lies in the union of these subsets. Any subset of a cover which is again a cover is called a *subcover*. A topological space X is *compact* if every open cover of X has a finite subcover.

Exercise 1.3.8. Show that if X is compact and Y is homeomorphic to X, then Y is compact as well.

Exercise 1.3.9. Show that if X is compact and $f \colon X \to Y$ is continuous, then $f(X)$ is compact.

Exercise 1.3.10. Show that every closed subset of a compact space is compact.

We say that a subset of X is *closed* if its complement is *open*.

Exercise 1.3.11. Show that a subset of $\mathbf{R}$ may be both open and closed. Also show that a subset of $\mathbf{R}$ may be neither open nor closed.

The n-dimensional Euclidean *ball* of radius r centered at p is defined by

$$B_r^n(p) := \{x \in \mathbf{R}^n \mid \operatorname{dist}(x, p) \leq r\}.$$

A subset A of $\mathbf{R}^n$ is *bounded* if $A \subset B_r^n(o)$ for some $r \in \mathbf{R}$. The following is one of the fundamental results of topology.

Theorem 1.3.12. *A subset of* $\mathbf{R}^n$ *is compact if and only if it is closed and bounded.*

The above theorem can be used to show:

Exercise 1.3.13. Show that $\mathbf{S}^n$ is not homeomorphic to $\mathbf{R}^m$.

Next, we show that $\mathbf{R}^2$ is not homeomorphic to $\mathbf{R}^1$. This can be done by using the notion of connectedness.

We say that a topological space X is *connected* if and only if the only subsets of X which are both open and closed are $\emptyset$ and X.

Exercise 1.3.14. Show that if X is connected and Y is homeomorphic to X then Y is connected.

Exercise 1.3.15. Show that if X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.

We also have the following fundamental result:

Theorem 1.3.16. $\mathbf{R}$ *and all of its intervals* $[a, b]$, (a, b) *are connected.*

We say that X is path connected if for every x_0, $x_1 \in X$, there is a continuous mapping $f\colon [0, 1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$.

Exercise 1.3.17. Show that if X is path connected and Y is homeomorphic to X then Y is path connected.

Exercise 1.3.18. Show that if X is path connected, then it its connected.

Exercise 1.3.19. Show that $\mathbf{R}^2$ is not homeomorphic to $\mathbf{R}^1$. (Hint: Suppose that there is a homeomorphism $f\colon \mathbf{R}^2 \to \mathbf{R}$. Then for a point $p \in \mathbf{R}^2$, f is a homeomorphism between $\mathbf{R}^2 - p$ and $\mathbf{R} - f(p)$.)

The technique hinted in Exercise 1.3.19 can also be used in the following:

Exercise 1.3.20. Show that the figure "8", with respect to its subspace topology, is not homeomorphic to $\mathbf{S}^1$.

Finally, we show that $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^m$ if $m \neq n$. This is a difficult theorem requiring homology theory; however, it may be proved as an easy corollary of the generalized Jordan curve thoerem:

Theorem 1.3.21 (Genralized Jordan). *Let* $X \subset \mathbf{R}^n$ *be homeomorphic to* $\mathbf{S}^n$ *(with respect to the subspace topology). Then* $\mathbf{R}^n - \mathbf{S}^n$ *is not connected.*

Use the above theorem to solve the following:

Exercise 1.3.22. Show that $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^m$ unles $m = n$.