Math 550 Apr 29, 2002
Vector Calculus
Spring 2002, USC

Review Problems

1. Prove

(a) $\|u \times v\|^2 = \|u\|^2\|v\|^2 - (u \cdot v)^2$

(b) The midpoints of a quadrilateral determine a parallelogram.

(c) The Pythagorean theorem.

(d) The diagonals of a prallelogram are orthogonal if and only if the parallelogram is a rhombus.

2. Find the distance between

(a) The point $(3, 4, 5)$ and the plane $2x + y + 3z = 5$.

(b) The lines $\ell_1(t) = t(8, -1, 0) + (-1, 3, 5)$ and $\ell_2(t) = t(0, 3, 1) + (0, 3, 4)$.

(c) The point $(2, -1)$ and the line $\ell\colon x = 3t + 7, y = 5t - 3$.

3. Evaluate

(a) $\int\int_D \sin(x^2 + y^2)dxdy$ where D is the disk $x^2 + y^2 \leq \pi$.

(b) $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

(c) $\int_0^{\pi} \int_y^{\pi} \frac{\sin x}{x}\, dx\, dy$

4. Find the center of mass of:

(a) The icecream cone given by $x^2 + y^2 + z^2 \leq 1$ and $z \geq \sqrt{x^2 + y^2}$ if the density is $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

(b) The tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$ (just set up the integrals).

5. Find the average value of:

(a) The y coordinate of the half-disk $x^2 + y^2 \leq 1$, $y > 0$.

(b) The z coordinate of the half-ball $x^2 + y^2 + z^2 \leq 1$, $z > 0$.

(c) The y coordinate of the semicircle $x^2 + y^2 = 1$, $y > 0$.

(d) The z coordinate of the hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$.

6. Use Greens theorem to compute the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

7. Show that the gravitational vector filed $\mathbf{F} := -\frac{\mathbf{r}}{\|\mathbf{r}\|^3}$ is conservative. What is the total work done in moving a particle from a point $\mathbf{r_0}$ to a point $\mathbf{r_1}$.

8. Use Gauss's theorem to show that the volume of the cone with base D and height h is given by $1/3Area(D)h$.

9. Show that the length of the graph of the function $y = f(x)$, $a \leq x \leq b$, is given by $\int_a^b \sqrt{1 + f'(x)^2}\, dx$.

10. Show that the area of a surface given by rotating the graph of the function $y = f(x)$, $a \leq x \leq b$, around x-axis is given by $2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2} dx$.

11. Use the previous problem to show that the area of a sphere cut by a pair of parallel planes depends only on the distance between the two planes.

12. Find $\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ where S is the hemisphere given by $x^2 + y^2 + z^2 = 1$ and $\mathbf{F}(x, y, z) := (x + z, y + z, z^2)$ (*Hint*: use Stokes theorem).

13. Suppose that a particle of mass m moves on a path $\mathbf{c}(t)$ in the gravitational vectorfield $\mathbf{F}$ according to Newton's second law: $\mathbf{F}(c(t)) = m\mathbf{c}''(t)$. Show that (a) the angular momentum $\mathbf{h}(t) := \mathbf{c}(t) \times \mathbf{c}'(t)$ stays constant in time, and (b) $\mathbf{c}(t) \cdot \mathbf{h}(t) = 0$. What can we conclude from (a) and (b) with regard to the path of the particle?

14. Suppose that a particle of mass m moves along a curve $\mathbf{c}(t)$, $a \leq t \leq b$, inside a vector field $\mathbf{F}$. Show that the total work $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{2}mv^2(b) - \frac{1}{2}mv^2(a)$, where $v(t) = \|\mathbf{c}'(t)\|$. (*Hint*: Use Newton's second law: $\mathbf{F}(c(t)) = m\mathbf{c}''(t)$.)

15. Show that there is no gravitational force inside a hollow planet.