Math 550 Sep 26, 2001
Vector Analysis
Fall 2001, USC

PRACTICE QUIZ 4

Basic Concepts

The Chain Rule:

$$\big(f\circ g\big)'(x) = f'\big(g(x)\big)\cdot g'(x).$$

The Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f'(t)\,dt = f(x).$$

The Substitution Rule:

$$\int_a^b f\big(u(x)\big)\frac{du}{dx}\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Basic Definitions

By a curve we mean a differentiable mapping $r\colon [a,b]\to\mathbf{R}^3$ with $\|r'\|\neq 0$. The length of r is given by

$$\mathrm{Length}[r] := \int_a^b \|r'(t)\|\,dt.$$

The arclength function is defined as

$$s(t) := \int_a^t \|r'(\tau)\|\,d\tau.$$

By a reparametrization of r, with respect to ϕ, we mean the curve $r\circ\phi$, where $\phi\colon [c,d]\to[a,b]$ is an onto function. When $\|r'\|=1$, we say that r is parametrized by arclength.

1. Show that when $\phi'(t) \neq 0$, then the length is independent of the reparametrization, i.e.,

$$\text{Length}[r \circ \phi] = \text{Length}[r].$$

Hints: (i) Apply the definition of length to the left hand side, (ii) use the chain rule, and then (iii) the substitution rule.

2. Use the chain rule to prove that if f is a differentiable function with differentiable inverse f^{-1}, then

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

Hints: (i) Note that $(f \circ f^{-1})(x) = x$, (ii) differentiate both sides, and (iii) solve for $(f^{-1})'(x)$.

3. Show that the arclength function s is invertible, and

$$(s^{-1})'(t) = \frac{1}{\|r'\big(s^{-1}(t)\big)\|}.$$

Hints: (i) Use the Fundamental Theorem of Calculus to compute $s'(t)$, (ii) conclude that s is one-to-one, and is thus invertible, because $s' > 0$, and then (iii) apply the result of the previous problem.

4. Show that if $\phi(t) := s^{-1}(t)$, then

$$\|\big(r \circ \phi\big)'(t)\| = 1.$$

Hints: Use the chain rule, and the result of the previous problem.

5. The previous problem shows that every curve may be reparametrized by arclength. Use the same procedure to reparametrize the helix

$$r(t) = (R\cos t, R\sin t, ht)$$

by arclength, where R and h are constants, and then compute the curvature of r (recall that when a curve is parametrized by arclength, then the curvature is given by $\|r''\|$).