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Geometry and Topology II
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Lecture Notes 9

3 Some Topics from Differential Topology

3.1 Regular points and values; Fundamental Theorem of Algebra

Let $f\colon M \to N$ be a smooth map. We say that $p \in M$ is a *regular point* of f provided that $rank(df_p) = dim(N)$; otherwise we say that p is a *critical point* or a *singular point*. We say $q \in N$ is a *regular value* of f provided that every $p \in f^{-1}(q)$ is a regular point. If q is not a regular value of f, then we say that it is a *critical value* or *singular* value.

Exercise 1. Show that if $f\colon M^m \to N^n$ has rank k at some point $p \in M$, then it has rank k on a neighborhood of p.

Proposition 2. *If $f\colon M^m \to N^n$ is a smooth map, q is a regular value of f, and $f^{-1}(q) \neq \emptyset$, then $f^{-1}(q)$ is a smooth $(m-n)$-dimensional submanifold of M.*

Proof. By definition, every $p \in f^{-1}(q)$ is a regular point. Thus, f has constant rank n on $f^{-1}(q)$. This implies that f has constant rank n on an open neighborhood of U of $f^{-1}(q)$. Since $f\colon U \to N$ has constant rank n, it now follows, as we had proved earlier, that $f^{-1}(q)$ is an $(m-n)$-dimensional submanifold of U, and therefore of M. □

Exercise 3. Show that if $f\colon M \to N$ is a smooth map, $dim(M) = dim(N)$, M is compact, and q is a regular value of f, then $f^{-1}(q)$ consists of a finite number of points. Further, show that if we denote the number of these points by $\#f^{-1}(q)$, then $\#f^{-1}(q)$ is locally constant, i.e., there is an open neighborhood U of q in N, such that $\#f^{-1}(q') = \#f^{-1}(q)$, for all $q' \in U$.

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The following proof demonstrates the elegant and powerful utility of basic techniques of Differential topology.

Theorem 4 (Fundamental Theorem of Algebra). *Every nonconstant complex polynomial has a zero.*

Proof(After Milnor). Let $P\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a complex polynomial, and $\pi_+\colon \mathbf{S}^2 - \{(0,0,1)\} \to \mathbf{R}^2$ be the stereographic projection from the north pole. Define $f\colon \mathbf{S}^2 \to \mathbf{S}^2$ by $f((0,0,1)) := (0,0,1)$, and

$$f(p) := (\pi_+)^{-1} \circ P \circ \pi_+(p),$$

if $p \neq (0,0,1)$.

We claim that f is smooth. This is obvious on $\mathbf{S}^2 - \{(0,0,1)\}$ where f is the composition of three smooth functions. To see that f is smooth in a neighborhood of $(0,0,1)$ as well, let $\pi_-\colon \mathbf{S}^2 - \{(0,0,-1)\} \to \mathbf{R}^2$ be the stereographic projection from the south pole, $U \subset \mathbf{R}^2$ be a small neighborhood of the origin $o \in \mathbf{R}^2$, and define $Q\colon U \to \mathbf{R}^2$ by

$$Q(z) := \pi_- \circ f \circ (\pi_-)^{-1}(z).$$

Note that if U is sufficiently small $f \circ (\pi_-)^{-1}(z)$ is close to $f \circ (\pi_-)^{-1}(o) = (0,0,1)$ for all $z \in U$. In particular, if U is sufficiently small, $f \circ (\pi_-)^{-1}(z) \neq (0,0,-1)$ for all $z \in U$. Thus Q is well defined. Secondly, note that $\pi_+ \circ (\pi_-)^{-1}$ is inversion with respect to the unit circle in $\mathbf{R}^2$, i.e., $\pi_+ \circ (\pi_-)^{-1}(z) = z/\|z\|^2 = 1/\overline{z}$. This yields that, if $P(z) = \sum_{i=0}^{n} a_i z^i$, with $a_n \neq o$, then

$$\begin{aligned} Q(z) &= \pi_- \circ (\pi_+)^{-1} \circ P \circ \pi_+ \circ (\pi_-)^{-1}(z) \\ &= ((\pi_+) \circ (\pi_-)^{-1})^{-1} \circ P(1/\overline{z}) \\ &= \frac{1}{\overline{P(1/\overline{z})}} \\ &= \frac{z^n}{\sum_{i=0}^{n} \overline{a}_i z^{n-i}}. \end{aligned}$$

Thus Q is smooth on U, which yields that f is smooth near $(0,0,1)$.

Next note that df_p is singular, if and only of $\pi_+(p)$ is a root of the complex polynomial $P'(z) = \sum_{i=1}^{n} a_i i z^{i-1}$ (see the next exercise). But a complex polynomial has only finitely many roots unless it is identically zero. So, since by assumption $a_n \neq o$, we conclude that the set of regular values of f

consists of $\mathbf{S}^2$ minus a finite number of points. In particular, the set of regular values of f is connected and open. So the locally constant function $\#f^{-1}$ is constant on the set of regular values of f. Since the number of singular points of f are finite, $\#f^{-1}$ cannot be zero everywhere, so it is zero nowhere on the set of regular values of f. This yields that f is onto. In particular, there exists $p \in \mathbf{S}^2$ such that $f(p) = (0, 0, -1)$. So $P(\pi_+(p)) = \pi_+(f(p)) = o$. □

Exercise 5. Let $P\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a complex polynomial. Show that

$$dP_z(w) = P'(z)w.$$

More precisely if $\theta_z\colon T_z\mathbf{R}^2 \to \mathbf{R}^2$ denotes the standard isomorphism, then $dP_z(w) = \theta_z^{-1}(P'(z)\theta_z(w))$. In particular z is a singular point of P if and only if z is a root of $P'(z)$

3.2 Manifolds with boundary

Next we generalize the last proposition, concerning the inverse image of regular values, to manifolds with boundary. Recall that M is a manifold with boundary if every point of M has an open neighborhood which is homeomorphic to an open subset of the half space

$$H^n := \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_n \geq 0\}.$$

We define the interior, $\operatorname{int} H^n$, and boundary, ∂H^n, as the subsets of H^n where $x^n > 0$ and $x^n = 0$ respectively. We say that p is an interior point of M if an open neighborhood of p is homeomorphic to an open neighborhood in $\operatorname{int} H^n$; otherwise, we say that p is a boundary point.

Exercise 6. Show that if M^n is an n-manifold with $\partial M \neq \emptyset$, then ∂M is an $(n-1)$-manifold without boundary.

Similar to the case of manifolds (without boundary), we say that a manifold with boundary is smooth if it admits an atlas such that all the local charts in that atlas are C^∞ compatible, i.e., if (U, ϕ) and (V, ψ) are local charts of M then $\phi \circ \psi^{-1}\colon \psi(U \cap V) \to \mathbf{R}^n$ is smooth. Further, it is important to recall that if $A \subset M$ is any subset, then $f\colon A \to N$ is smooth provided that for every $p \in A$ there exists an open neighborhood U of p in M and a smooth function $f_p\colon U \to N$ such that $f|_{U \cap A} = f_p$. For any $p \in M$, we may define T_pM as the equivalence class of all the half curves which originate or end at p. More explicitly, define $Curves_pM$ as the set of curves $\alpha\colon [0,\epsilon) \to M$ and $\alpha\colon (-\epsilon, 0] \to M$ such $\alpha(0) = p$. We say that a pair of curves α, $\beta \in Curves_pM$ are equivalent provided that there exists a local chart (U, ϕ) centered at p such that $(\phi \circ \alpha)'(0) = (\psi \circ \beta)'(0)$. Then T_pM is defined as the collection of equivalence classes $Curves_pM/\sim$.

Exercise 7. Check that the above definition for T_pM coincides with the one we had given earlier whenever $p \in \operatorname{int} M$.

Exercise 8. Show that if M^n is a manifold with boundary, $p \in M$, and (U, ϕ) is any local chart centered at p, then the mapping

$$T_pM \ni [\alpha] \longmapsto (\phi \circ \alpha)'(0) \in \mathbf{R}^n$$

is a bijection. Thus we may use this map to endow T_pM with the structure of an n-dimensional vector space.

The differential of any smooth map $f\colon M \to N$, where M is a manifold with boundary, is defined as before.

Exercise 9. Let $f\colon H^m \to M$ be a smooth map and $p \in \partial H$. Show that if U is any open neighborhood of p in $\mathbf{R}^n$ and $\tilde{f}\colon U \to M$ is any smooth map such that $\tilde{f} = f$ on $U \cap H^m$, then $df_p = d\tilde{f}_p$.

Exercise 10. Let M^m be a smooth manifold with boundary and $f\colon M^m \to \mathbf{R}$ be a smooth map which has 0 as a regular value. Then $f^{-1}([0,\infty))$ is a smooth m-manifold with boundary and $\partial f^{-1}([0,\infty)) = f^{-1}(0)$.

Our main aim in this section is to prove:

Theorem 11. *Let M^m be a manifold with boundary and $f\colon M^m \to N^n$ be a smooth map. Suppose that $q \in N$ is a regular value of both f and $f|_{\partial M}$, and $f^{-1}(q) \neq \emptyset$, then $f^{-1}(q)$ is a smooth $m-n$ manifold with boundary, and*

$$\partial\big(f^{-1}(q)\big) = f^{-1}(q) \cap \partial M.$$

To prove the above theorem we need a couple of more exercises:

Exercise 12. Show that to prove the above theorem it is enough to consider the case of $M^m = H^m$.

Exercise 13. Show that if M is a manifold (without boundary) and $f\colon M \to N$ is a smooth map which has q as a regular value, then, for every $p \in f^{-1}(q)$, $T_p f^{-1}(q)$ is the null space of df_p.

Proof of Theorem 11. We know that $f^{-1}(q) \cap \operatorname{int} M$ is a smooth manifold, because $\operatorname{int} M$ is a manifold (without boundary) and q is a regular value of $f|_{\operatorname{int} M}$. Thus it remains to consider points $p \in f^{-1}(q) \cap \partial M$.

We may suppose $M = H^m$. Then, by definition of smoothness, there exists, for every $p \in \partial M = \partial H$, an open neighborhood V of p in $\mathbf{R}^m$ and a smooth function $\tilde{f}\colon V \to N$ such that $\tilde{f} = f$ on $V \cap H^m$. Since, by a previous exercise, $d\tilde{f}_p = df_p$, it follows that p is a regular point of $\tilde{f}$. Thus, after replacing V by a smaller neighborhood of p, which we again denote by V, we may assume that $\tilde{f}$ as no critical points in V. In particular, it follows that q is a regular value of $\tilde{f}$. So $\tilde{f}^{-1}(q)$ is a smooth submanifold of V. Now define $g\colon \tilde{f}^{-1}(q) \to \mathbf{R}$ by $g(x_1, \dots, x_n) = x_n$. Then $g(p) = 0$ and $V \cap f^{-1}(q) = H \cap \tilde{f}^{-1}(q) = g^{-1}([0, \infty))$. Thus to complete the proof it suffices to show that 0 is a regular value of g. Suppose not. Then $T_p\tilde{f}^{-1}(q)$ is equal to the null space of dg_p. But the null space of dg_p is a subset of $T_p\partial H$. Thus, since $T_p\tilde{f}^{-1}(q)$ and $T_p\partial H$ have the same dimension, it follows that $T_p\tilde{f}^{-1}(q) = T_p\partial H$. But $T_p\tilde{f}^{-1}(q)$ is the null space of $d\tilde{f}_p$ which is equal to the null space of df_p. So the null space of df_p is equal to $T_p\partial H$, which contradicts the assumption that p is a regular point of $f|_{\partial H}$. □

3.3 Sard's Theorem, and Brouwer's Fixed Point Theorem

Let $f\colon M \to N$ be a smooth map. Sard's theorem states that almost every point $q \in N$ is a regular value of f, where "almost every", or "almost all", means except for a set of measure zero. This theorem, whose proof we postpone for the time being, has great many applications, including the Brouwer's fixed point theorem which we prove below.

Exercise 14. Let $\Gamma \subset \mathbf{R}^2$ be a smooth simple closed curve, i.e., the image of a smooth embedding of $\mathbf{S}^1$. For any unit vector $u \in \mathbf{S}^1$, the height function $h_u\colon \Gamma \to \mathbf{R}$ is defined as $h_u(p) = \langle p, u\rangle$. Use Sard's theorem to show that, for almost all $u \in \mathbf{S}^2$, h_u has a finite number of critical points (*Hint:* It is enough to show that for almost every $u \in \mathbf{S}^1$ there exist only finitely many tangent lines of Γ which are orthogonal to u).

Our main aim in this section is to use Sard's theorem to show that

Theorem 15 (Brouwer). *For $n \geq 2$, any continuous map $f\colon B^n \to B^n$ has a fixed point, where B^n denotes the n-dimensional closed unit ball in $\mathbf{R}^n$.*

The proof is by contradiction and requires the following lemmas:

Lemma 16. *If there exists a continuous map $f\colon B^n \to B^n$ without fixed points, then there exists a* smooth *map $\tilde{f}\colon B^n \to B^n$ without fixed points.*

Proof. If f has no fixed points, then, since B^n is compact, there exists an $\epsilon > 0$ such that $\|f(p) - p\| > \epsilon$ for all $p \in B^n$. By Wierstrauss approximation theorem, there exists a smooth map $\overline{f}\colon B^n \to \mathbf{R}^n$ such that $\|\overline{f}(p) - f(p)\| < \epsilon/2$ for all $p \in B^n$. Let

$$\tilde{f} := \frac{1}{1 + \epsilon/2}\overline{f}.$$

Then $\tilde{f}\colon B^n \to B^n$, because, by the triangle inequality,

$$\|\overline{f}(p)\| \leq \|f(p)\| + \|\overline{f}(p) - f(p)\| < 1 + \epsilon/2,$$

which yields that $\|\tilde{f}\| \leq 1$. Further note that, again by the triangle inequality,

$$\|\tilde{f}(p) - f(p)\| \leq \|\tilde{f}(p) - \overline{f}(p)\| + \|f(p) - \overline{f}(p)\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, since $\|f(p) - p\| > \epsilon$, it follows that

$$\|\tilde{f}(p) - p\| \geq \|f(p) - p\| - \|\tilde{f}(p) - f(p)\| > 0.$$

So $\tilde{f}$ has no fixed points. □

Lemma 17. *If there exists a smooth map $f\colon B^n \to B^n$ without fixed points, then there exists a smooth map $r\colon B^n \to \mathbf{S}^{n-1}$ such that $r(p) = p$ for all $p \in \mathbf{S}^{n-1}$.*

Proof. Consider the ray which starts from $f(p)$ and passes through p. Let $r(p)$ be the intersection of this ray with $\mathbf{S}^{n-1}$. To see that r is smooth, note that the ray may be parametrized by

$$\ell(t) := f(p) + t\big(p - f(p)\big),$$

where $t \geq 0$. Solving $\|\ell(t)\| = 1$ for t and substituting the solution back in $\ell(t)$ gives an explicit expression for $r(p)$, which one may check to be smooth. □

Exercise 18. Find the explicit expression for r in the above lemma and show that r is smooth.

Now we are ready to prove the main result of this section:

Proof of Brouwer's Theorem. Let r be as in the previous lemma. By Sard's theorem, r has a regular value $p \in \mathbf{S}^{n-1}$. Then $r^{-1}(p)$ is a 1-dimensional manifold with boundary which contains p, since $r(p) = p$. Further recall that $\partial r^{-1}(p) = r^{-1}(p) \cap \mathbf{S}^{n-1}$. But $r^{-1}(p) \cap \mathbf{S}^{n-1} = \{p\}$ because r is one-to-one on $\mathbf{S}^{n-1}$. Thus $\partial r^{-1}(p) = \{p\}$. But $r^{-1}(p)$, being the closed subset of a compact space, is compact, and each component of a compact 1-dimensional manifold with boundary must have either zero or two boundary points. So we have a contradiction. $\square$

Note that the above proof uses the fact that every compact connected one dimensional manifold with boundary is either homeomorphic to $\mathbf{S}^1$ or the interval $[0, 1]$. Can you prove this fact?