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Geometry and Topology II
Spring 2005, PSU

Lecture Notes 8

2.10 Measure of C^1 maps

If X is a topological space, we say that $A \subset X$ is *dense* is X provided that $\overline{A} = X$, where $\overline{A}$ denotes the closure of A. In other words, A is dense in X if every open subset of X intersects A.

Theorem 1. *Let $f: M^n \to N^m$ be a C^1 map. Suppose that $m > n$. Then $N - f(M)$ is dense in M.*

To prove the above result we need to develop the notion of measure zero, which is defined as follows. We say that $C \subset \mathbf{R}^n$ is a cube of side length λ provided that

$$C = [a_1, a_1 + \lambda] \times \cdots \times [a_n, a_n + \lambda],$$

for some $a_1, \ldots, a_n \in \mathbf{R}^n$. We define the mesure or volume of a cube of side length λ by

$$\mu(C) := \lambda^n.$$

We say a $X \subset \mathbf{R}^n$ has *measure zero* if for every $\epsilon > 0$, we may cover X by a family of cubes C_i, $i \in I$, such that $\sum_{i \in I} \mu(C_i) \leq \epsilon$.

Lemma 2. *A countable union of sets of measure zero in $\mathbf{R}^n$ has measure zero.*

Proof. Let X_i, $i = 1, 2, \ldots$ be a countable collection of subsets of $\mathbf{R}^n$ with measure zero. Then we may cover each X_i by a family C_{ij} of cubes such that $\sum_j C_{ij} < \epsilon/2^i$. Then

$$\sum_{i=1}^{\infty} \sum_j C_{ij} = \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon.$$

Since $\cup_i X_i \subset \cup_{ij} C_{ij}$, it follows then that $\cup_i X_i$ has measure zero. □

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Lemma 3. *If $U \subset \mathbf{R}^n$ is open and nonemepty, then it cannot have measure zero.*

Proof. By definition, for evey point $p \in U$ there exists $r > 0$ such that $B_r(p) \subset U$. Thus, since $U \neq \emptyset$, U contains a cube C (of side length $\lambda \leq 2r/\sqrt{n}$). Suppose there is a covering of U by a family of cubes. Then, since C is compact, there exists a finite subcollection C_i, $i = 1 \ldots, m$ which cover C. Let N be the number of integer lattice points (i.e., points with integer coefficients) which lie in C, then

$$\big(\max(0, \lambda - 1)\big)^n \leq N \leq (\lambda + 1)^n.$$

Similarly, if $N - i$ is the number of integer lattice points in C_i and and C_i has edge length λ_i, then

$$\big(\max(0, \lambda_i - 1)\big)^n \leq N_i \leq (\lambda_i + 1)^n.$$

Now note that, since C_i cover C, that $N \leq \sum_{i=1}^m N_i$. Thus,

$$\big(\max(0, \lambda - 1)\big)^n \leq \sum_{i=1}^m (\lambda_i + 1)^n.$$

Next note that if we scale all the cubes by a factor of k, and let N^k, N_i^k denote the number of lattice point in kC and kC_i respectively, we still have $N^k \leq \sum_{i=1}^m N_i^k$. Thus it follows that

$$\big(\max(0, k\lambda - 1)\big)^n \leq \sum_{i=1}^m (k\lambda_i + 1)^n,$$

for any $k > 0$. In particular, assuming $k \geq 1/\lambda$, we have

$$(k\lambda - 1)^n \leq \sum_{i=1}^m (k\lambda_i + 1)^n,$$

which in turn yields

$$\left(\lambda - \frac{1}{k}\right)^n \leq \sum_{i=1}^m \left(\lambda_i + \frac{1}{k}\right)^n.$$

Taking the limit of both sides as $k \to \infty$ yields

$$\lambda^n \leq \sum_{i=1}^{m} \lambda_i^n = \sum_{i=1}^{m} \mu(C_i).$$

Thus the total measure of any covering of U by cubes is bounded below by a positive constant, and therefore, U cannot have measure zero. □

Lemma 4. *Let $U \subset \mathbf{R}^n$ be an open subset and $f\colon U \to \mathbf{R}^n$ be a C^1 map. Suppose that $X \subset U$ has measure zero. Then $f(X)$ has measure zero.*

Proof. Define $K\colon U \to \mathbf{R}$ by

$$K(p) := \max(D_i f^j(p)),$$

where $1 \leq i, j \leq n$. Then, since f is C^1, for each $p \in U$, there exists an open neighborhood V_p of p in U such that

$$\max(D_i f^j(q)) \leq K(p) + 1, \quad \text{for all} \quad q \in V_p.$$

In particular we may let V_p be a small ball with rational radius centered at a point with rational coefficients. So there exists a countable family of open neighborhoods V_ℓ which cover U_ℓ such that

$$\max(D_i f^j(q)) \leq K_\ell, \quad \text{for all} \quad q \in V_\ell.$$

So it follows by a lemma we proved earlier (which was a consequence of the mean value theorem) that

$$\|f(p) - f(q)\| \leq K_\ell \|p - q\| \quad \text{for all} \quad p, q \in V_\ell.$$

Now let $X_\ell := X \cap V_\ell$. Then $f(X) = \cup_\ell f(X_\ell)$. In particular, since $f(X_\ell)$ is countable, to prove that $f(X)$ has measure zero, it suffices to show that each $f(X_\ell)$ has measure zero.

To see that $f(X_\ell)$ has measure zero, forst note that X_ℓ has measure zsro, since it is a subset of X which has measure zero by assumption. Thus we may cover X_ℓ be a colloection C_i of cubes of total measure less than ϵ, for any $\epsilon > 0$. Now note that each C_i is contained in a ball of radius $\sqrt{n}\lambda_i/2$, where λ_i is the edge length of C_i. Thus $f(C_i)$ is contained in a ball of radius

$L\lambda_\ell/2$, where $L := K_\ell\sqrt{n}$, which yields that $f(C_i)$ is contained in a cube C_i' of edge length $L\lambda_\ell$. So

$$\sum_{i=1}^{\infty}\mu(C_i') = \sum_{i=1}^{\infty} L^n\lambda_\ell^n = L^n\sum_{i=1}^{\infty}\mu(C_i) \leq L^n\,\epsilon.$$

Since L does not depend on ϵ, and C_i' cover $f(X_\ell)$ we conclude then that $f(X_\ell)$ has measure zero. □

We say a $X \subset M$ has measure zero provided that for every $p \in M$ there exists a local chart (U, ϕ) such that $\phi(U \cap X)$ has measure zero. The last result can be used to show that this concept is well defined:

Exercise 5. Show that the concept of measure zero for a subset of a manifold does not depent on the choice of local charts.

Further, the earlier result that open subsets of $\mathbf{R}^n$ cannot have measure zero, can be used in the following:

Exercise 6. Show that if a $X \subset M$ has measure zero, then $M - X$ is dense.

Thus to prove Theorem 1, we just need to show that $f(M)$ has measure zero in N. To this end, we first show:

Lemma 7. *If $U \subset \mathbf{R}^n$ is open, $f\colon U \to \mathbf{R}^m$ is C^1, and $m \geq n$, then $f(U)$ has measure zero in $\mathbf{R}^m$.*

Proof. Let $\pi\colon \mathbf{R}^m \to \mathbf{R}^n$ be the projection into the first n coordinates. Then $f \circ \pi\colon \pi^{-1}(U) \to \mathbf{R}^m$ is C^1. Thus, since U has measure zero in $\pi^{-1}(U)$, it follows that $f \circ \pi(U)$ has measure zero. But $f \circ \pi(U) = f(U)$, so we are done. □

Now we are ready to prove Theorem 1. This proof requires the following facts.

Exercise 8. Show that every manifold admits a countable Atlas.

Exercise 9. Show that a countable union of sets of measure zero in a manifold has measure zero.

Proof of Theorem 1. Let (U_i, ϕ_i) be a countable atlas for M. Since $f(U_i)$ covers $f(M)$, it suffices to show that $f(U_i)$ has measure zero in N. To see this let $p \in f(U_i)$, and (V, ψ) be a local chart of N centered at p. Then $\psi(V \cap f(U_i)) = \psi(V \cap f(\phi_i^{-1}(\mathbf{R}^n))) \subset \psi \circ f \circ \phi^{-1}(\mathbf{R}^n)$ which has measure zero in $\mathbf{R}^m$. □

2.11 Whitney's 2n+1 Embedding Theorem

Here we show that

Theorem 10. *Every smooth compact manifold M^n admits a smooth embedding into R^{2n+1}.*

The basic idea for the proof of the above theorem is to embedd M^n is some Euclidean space $\mathbf{R}^N$ (which, as we have already shown, is possible for N sufficiently large) and then reduce the codimension $(N-n)$ by successive projections. More precisely, let $f\colon M \to \mathbf{R}^N$ be an embedding, identify M with $f(M)$, and for $u \in \mathbf{S}^{N-1}$, define $\pi\colon \mathbf{R}^N \to \mathbf{R}^{N-1}$ by

$$\pi_u(x) := x - \langle x, u\rangle u.$$

We claim that if $N > 2n+1$, then there exists $u \in \mathbf{S}^{N-1}$ such that $\pi_u|_M$ is an embedding, which would complete the proof. To establish the claim recall that all we need is to find a u such that (i) $\pi_u|_M$ is one-to-one and (ii) $\pi_u|_M$ has full rank.

In order to meet condition (i), we proceed as follows. Let

$$\Delta_M := \{(p,p) \mid p \in M\},$$

be the *diagonal* of $M \times M$. Note that since Δ_M is closed in $M \times M$, $M \times M - \Delta_M$ is an open subset of $M \times M$ and is therefore a $2n$ dimensional manifold. Now define $\sigma\colon M \times M - \Delta_M \to \mathbf{S}^{N-1}$ by

$$\sigma(p,q) := \frac{p-q}{\|p-q\|},$$

and note that , if $N > 2n+1$, then $\dim(M \times M - \Delta_M) = 2n < N-1 = \dim(\mathbf{S}^{N-1})$. Thus, by the result in the previous subsection, the image of σ has measure zeo in $\mathbf{S}^{N-1}$. In particular, there exists $u \in \mathbf{S}^{N-1}$ such that $u \notin \pm\sigma(M \times M - \Delta_M)$. Then π_u is one-to-one, because if $\pi_u(p) = \pi_u(q)$, we have $p - q = \langle p-q, u\rangle u$, which yields that $p-q$ is either parallel or antiparallel to u. Thus, since $\|u\| = 1$, it would follow that either $\sigma(p,q) = u$ or $\sigma(p,q) = -u$, which is not possible.

In order to find u such that $\pi_u|_M$ has full rank note that, since $M \subset \mathbf{R}^N$, $T_pM \subset T_p\mathbf{R}^N$, for all $p \in M$. Thus, if $\theta - p\colon T_p\mathbf{R}^N \to \mathbf{R}^N$ is the standard isomorphism, $\theta_p(T_pM)$ is well-defined and is a subspace of $\mathbf{R}^N$. So, by an abuse of notation, we may identify T_pM with $\theta_p(T_pM)$.

Exercise 11. Show that $d(\pi_u)$ is nonsingluar at $p \in M$, if and only if $u \notin T_pM$.

So to complete the proof it suffices to show that the set of $u \in \mathbf{S}^{N-1}$ such that $u \in T_pM$ for some $p \in M$ has measure zero (since the union of two sets of measure zero has measure zero, we will then be able to find u such that π_u is one-to-one and is an immesion at the same time). To see this note that if we identify T_pM with $\theta_p(T_pM)$, then the tangent bundle T^1M gets identified with a sebset of S^{N-1} via a C^1 map. Thus, since as we showed earlier, T^1M has dimension $2n-1$, it follows that T^1M has measure zero in $\mathbf{S}^{N-1}$. So $TM \cap \mathbf{S}^{N-1}$ has measure zero, which completes the proof.