Math 598 — Feb 14, 2005[1]
Geometry and Topology II
Spring 2005, PSU

Lecture Notes 7

2.7 Smooth submanifolds

Let N be a smooth manifold. We say that $M \subset N^m$ is an n-dimensional *smooth submanifold* of N, provided that for every $p \in M$ there exists a local chart (U, ϕ) of N centered at p such that

$$\phi(U \cap M) = \mathbf{R}^n \times \{o\},$$

where o denotes the origin of $\mathbf{R}^{m-k}$.

Proposition 1. *A smooth submanifold $M \subset N$ is a smooth manifold.*

Proof. Since $M \subset N$, M is Hausdorf and has a countable basis. For every $p \in M$, let (U, ϕ) be a local chart of M with $\phi(U \cap M) = \mathbf{R}^n \times \{o\}$. Set $\overline{U} := U \cap M$, and $\overline{\phi} := \phi|_{\overline{U}}$. Then $\overline{\phi}\colon \overline{U} \to \mathbf{R}^n \times \{o\} \simeq \mathbf{R}^n$ is a homeomorphism, and thus M is a toplogical manifold. It remains to show that M is smooth. To see this note that if $(\overline{V}, \overline{\psi})$ is the restriction of another local chart of N to M. Then $\overline{\psi} \circ (\overline{\phi})^{-1} = \psi \circ \phi^{-1}|_{\overline{\phi}(\overline{U})}$, which is smooth. □

The above proof shows how M induces a differential structure on N. Whenver we talk of a submanifold M as a smooth manifold in its own right, we mean that M is equipped with the differential structue which it inherits from N.

Theorem 2. *Let $f\colon M^n \to N^m$ be a smooth map of constant rank k (i.e., rank$(df_p) = k$, for all $p \in M$). Then, for any $q \in N$, $f^{-1}(q)$ is an $(n-k)$-dimensional smooth submanifold of M.*

Proof. Let $p \in f^{-1}(q)$. By the rank theorem there exists local neighborhoods (U, ϕ) and (V, ψ) of M and N centered at p and q respectively such that

$$\tilde{f}(x) := \psi \circ f \circ \phi^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^k, 0, \dots, 0).$$

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Next note that

$$\phi(U \cap f^{-1}(q)) = \phi(U) \cap \phi \circ f^{-1} \circ \psi^{-1}(o) = \mathbf{R}^n \cap \tilde{f}^{-1}(o) = \{o\} \times \mathbf{R}^{n-k}.$$

Thus $f^{-1}(q)$ is a smooth submanifold of N (To be quite strict, we need to show that $\phi(U \cap f^{-1}(q)) = \mathbf{R}^{n-k} \times \{o\}$, but this is easily achieved if we replace ψ with $\theta \circ \psi$, where $\theta\colon \mathbf{R}^m \to \mathbf{R}^m$ is the diffeomorphism which switches the first k and last $m-k$ cordinates). $\square$

Exercise 3. Use the previous result to show that $\mathbf{S}^n$ is smooth n-dimensional submanifold of $\mathbf{R}^{n+1}$.

Another application of the last theorem is as follows:

Example 4. SL_n is a smooth submanifold of GL_n. To see this define $f\colon GL_n \to \mathbf{R}$ by $f(A) := \det(A)$. Then $SL_n = f^{-1}(1)$, and thus it remains to show that f has constant rank on GL_n. Since this rank has to be either 1 or 0 at each point (why?), it suffices to show that the rank is not zero anywhere, i.e., it is enough to show that for every $A \in GL_n$ there exists $X \in T_A GL_n$ such that $df_A(X) \neq 0$. To see this, let $X = [\alpha]$ where $\alpha\colon (-\epsilon, \epsilon) \to GL_n$ is the curve given by $\alpha(t) := (1-t)A$. Note that, since det is contiuous, $\det(\alpha(t)) \neq 0$, for all $t \in (-\epsilon, \epsilon)$, once we make sure that ϵ is small enough. Thus α is indeed well-defined. Now recall that

$$df_A(X) := [f \circ \alpha] \in T_{f(A)}\mathbf{R}.$$

Further recall that there is a canonical isomorphism $\theta\colon T_{f(A)}\mathbf{R} \to \mathbf{R}$ given by $\theta([\gamma]) = \gamma'(0)$. Thus

$$\theta \circ df_A(X) = (f \circ \alpha)'(0) = \det(A) \neq 0.$$

So, since θ is an isomorphism, $df_A(X) \neq 0$, as desired.

Exercise 5. Show that O_n is a smooth $n(n-1)/2$-dimensional submanifold of GL_n. (*Hint:* Define $f\colon GL_n \to GL_n$ by $f(A) := A^T A$. Then show that $T_A GL_n$ is given by the equivalence class of curves of the form $A + tB$ where B is any $n \times n$ matrix. Finally, show that $df_A(T_A GL_n)$ is isomorphic to the space of symmetric $n \times n$ matrices).

Note that if $A \in O_n$, the $\det(A) = \pm 1$. Thus O_n has two components. The component with positive determinant is known as the special orthogonal group SO_n.

Exercise 6. Show that SO_3 is diffeomorphic to $\mathbf{RP}^3$.

2.8 Smooth immersions and embeddings

We say, $f\colon M \to N$ is a *smooth embedding* if $f(M)$ is a smooth submanifold of N and $f\colon M \to f(M)$ is a diffeomorphism. If $f\colon M \to N$ is a local smooth embedding, i.e., every $p \in M$ has an open neighborhood U such that $f\colon U \to N$ is a smooth embedding, we say that f is a smoothe immersion.

Theorem 7. *Let $f\colon M^n \to N$ be a smooth map. Then f is an immersion if and only if f has constant rank n on M.*

Proof. If f is an immersion, then it is obvious from the definition of immersion (and the chain rule), that f has everywhere full rank (becuase then, locally, $f \circ f^{-1}$ is the identity map).

Conversely, suppse that f has constan rank n. Then, by the rank theorem, for every $p \in M$, there exists local charts (U, ϕ) and (V, ψ) of M and N centered at p and $f(p)$ respectively such that

$$\tilde{f}(x) = \psi \circ f \circ \phi^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0).$$

So f is one-to-one. Thus (since M is locally compact, and N is hausdorf) f is a local homeomoprhim. In particular, after replacing U by a smaller open neighborhood of p which has compact closure inside U, we may assume that $f^{-1}\colon f(U) \to U$ is well defined and continuous (we can always perform such a shrinking of U, since U is homeomoprphic to $\mathbf{R}^n$; in particular, we may replace U by $\phi^{-1}(\operatorname{int} B_1(o))$). Next note that

$$\psi(V \cap f(U)) = \psi(f(U)) = \tilde{f}(\phi(U)) = \tilde{f}(\mathbf{R}^n) = \mathbf{R}^n \times \{o\}.$$

Thus $f(U)$ is a smoooth submanifold of N. It remains, therefore, only to show that f^{-1} is smooth. To this end note that $\tilde{f}^{-1}\colon \mathbf{R}^n \times \{o\} \to f(U)$ is well defined. Further, since, as we showed above, $\psi\colon f(U) \to \mathbf{R}^n \times \{o\}$, it follows that

$$\phi^{-1} \circ \tilde{f}^{-1} \circ \psi = f^{-1}$$

on U. Since each of the maps on the left hand side of the above equation is smooth, f^{-1} is smooth. □

The following corollaries of the above theorem are immediate:

Corollary 8. *Let $f\colon M \to N$ be a smooth map. Then f is a smooth embedding if and only if f is a homeomorphism onto its image and f has full rank.*

Corollary 9. *Let $f\colon M \to N$ be a smooth map, and suppose that M is compact. Then f is a smooth embedding if and only if f is one-to-one and has full rank.*

Next we are going to use the last corollary to show that

Theorem 10. *Every smooth compact manifold M^n may be smoothly embedded in a Euclidean space.*

The proof of the above is a refinement of the proof we had given earlier for the existence of topological embeddings in Euclidean space. First we need to prove the following basic fact:

Lemma 11 (Existence of the smooth step function). *For any $0 < a < b$ there exits a smooth function $\sigma_{a,b}\colon \mathbf{R} \to \mathbf{R}$ such that $\sigma_{a,b} = 0$ on $(-\infty, r_1]$ and $\sigma_{a,b} = 1$ on $[r_2, \infty)$.*

Proof. Define $\phi\colon \mathbf{R} \to \mathbf{R}$ by

$$\phi(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases}$$

Then ϕ is smooth. Next define $\theta\colon \mathbf{R} \to \mathbf{R}$ by

$$\theta(x) := \phi(x-a)\phi(b-x).$$

Then θ is smooth, $\theta > 0$ on (a,b), and $\theta = 0$ on $(-\infty, a] \cup [b, \infty)$. Finally set

$$\sigma_{a,b}(x) := \frac{\int_x^a \theta(x)\,dx}{\int_a^b \theta(x)\,dx}.$$

□

Exercise 12. Show that the function ϕ in the above lemma is smooth.

Now we are ready to prove the main result of this section.

Proof of Theorem 10. As we had argued earlier, since M is compact, there exists a finite cover U_i, $1 \leq i \leq m$, of M and homeomorphisms $\phi_i\colon U_i \to \mathbf{R}^n$, such that $V_i := \phi_i^{-1}(\operatorname{int} B^n(1))$ also cover M.

Now define $\lambda\colon \mathbf{R}^n \to \mathbf{R}$ by $\lambda(x) := \sigma_{1,2}(\|x\|)$, where σ is the step fucntion defined above. Since $\|\cdot\|$ is smooth on $\mathbf{R}^n - \{o\}$ and λ is constant on an open neighborhood of o, it follows that that λ is smooth. In particular note that $\lambda = 1$ on $B^n(1)$ and $\lambda = 0$ on $\mathbf{R}^n - B^n(2)$. So if we define $\lambda_i \colon M \to \mathbf{R}$ by settting $\lambda_i(p) := \lambda(\phi_i(p))$ in case $p \in U_i$ and $\lambda_i(p) := 0$ otherwise, then $\lambda_i = 1$ on V_i and $\lambda_i = 0$ on $M - \phi_i^{-1}(B^n(2))$. In addition, we claim that, since M is hausorf, λ_i is smooth. To see this, let $K_i := \phi_i^{-1}(B^n(2))$. Then K_i is compact. So K_i is closed, since M is hausdorf. This yields that $M - K_i$ is open. In particular, since $K_i \subset U_i$, $\{U_i, M - K_i\}$ is an open cover of M. Since λ_i is smooth on U_i (where it is the composition of smooth functions) and λ_i is smooth on $M - K_i$ (where $\lambda_i = 0$) it follows that λ_i is smooth.

Next define $f_i \colon M \to \mathbf{R}^n$ by $f_i(p) = \lambda_i(p)\phi_i(p)$ if $p \in U_i$, and $f_i(p) = 0$ otherwise. Then f_i is smooth, since, similar to the argument we gave for λ_i above, f_i is smooth on U_i and $M - K_i$. Finally, define $f \colon M \to \mathbf{R}^{m(n+1)}$ by

$$f(p) = \big(\lambda_1(p), \dots, \lambda_m(p), f_1(p), \dots, f_m(p)\big).$$

Since each component function of f is smooth, f is smooth. We claim that f is the desired embedding. To this end, since f is smooth, and M is compact, it suffices to check that f is one-to-one and is an immersion.

To see that f is an immersion, note that, since V_i cover M and $\lambda_i = 1$ on V_i, at least one component function f_i is a diffeomorphism of a neighborhood of p into $\mathbf{R}^n$, and so has rank n at p. This implies that the rank of f is at least n, which since $\dim(M) = n$, implies in turn that rank of f is equal to n.

To see that f is one-to-one, suppose that $f(p) = f(q)$. Then $f_i(p) = f_i(q)$, and $\lambda_i(p) = \lambda_i(q)$. Since V_i cover M, $p \in V_j$ for some fixed j. Consequently

$$\lambda_j(q) = \lambda_j(p) \neq 0,$$

which yields that $q \in U_j$. Since $p, q \in U_j$, it follows, from definition of f_i, that

$$\lambda_j(p)\phi_j(p) = f_j(p) = f_j(q) = \lambda_j(q)\phi_j(q).$$

So we conclude that $\phi_j(p) = \phi_j(q)$, which yields that $p = q$. $\square$

2.9 Tangent bundle

If M^n is a smooth manifold then we set

$$TM := \bigcup_{p \in M} T_pM.$$

Note that if $X \in TM$, then $X \in T_pM$ for a unique $p \in M$. This defines a natural projection $\pi\colon TM \to M$.

Recall that for each $p \in \mathbf{R}^n$ there exists a canonical isomorphism $\theta_p\colon T_p\mathbf{R}^n \to \mathbf{R}^n$ (given by $\theta_p[\alpha] := \alpha'(0)$). Using this we may define a bijection $\theta\colon T\mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ by setting:

$$\theta(X) := \big(\pi(X), \theta_{\pi(X)}(X)\big).$$

We topologize $T\mathbf{R}^n$ by declaring that θ is a homeomorphism, i.e., we say that $U \subset T\mathbf{R}^n$ is open if and only if $\theta(U)$ is open. Further, we may use θ to endow $T\mathbf{R}^n$ with the standard differential structure of $\mathbf{R}^n$. Thus $T\mathbf{R}^n$ is a smooth $2n$-dimensional manifold.

Next note that if $f\colon M \to N$ is smooth then we may define a mapping $df\colon TM \to TN$ by setting $df|_{T_pM} := df_p$. If f is a diffeomorphism, then df is a bijection. Thus if (U, ϕ) is a local chart of M, then we obtain a bijection from TU to $\mathbf{R}^n \times \mathbf{R}^n$ given by

$$\theta_\phi(X) := \Big(\phi(p), \theta_{\phi(p)}\big(d\phi(X)\big)\Big), \quad \text{where} \quad p := \pi(X).$$

Requiring θ to be a homeomorphism topologizes TM. More explicitly, note that if (U_i, ϕ_i) is an atlas for M, then TU_i cover TM. We say that $V \subset TM$ is open if $\theta_{\phi_i}(V \cap TU_i)$ is open for every i. We define the *tangent bundle* of M as TM endowed with this topology. In particular, (TU_i, θ_{ϕ_i}) is an atlas for TM, and thus TM is a $2n$-manifold, once we check that:

Exercise 13. Show that TM is hausdorff and has a countable basis.

Furthermoe we can show:

Proposition 14. *If M^n is a smooth manifold, then TM is a smooth $2n$-manifold.*

Proof. It remains only to verify that the local charts (TU_i, θ_{ϕ_i}) are compatible, i.e., $\theta_{\phi_i} \circ \theta_{\phi_j}^{-1}$ is smooth (whenever $TU_i \cap TU_j \neq \emptyset$). To see this let $(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$, and suppose that

$$X := \theta_{\phi_j}^{-1}(x, y).$$

Then $\theta_{\phi_j}(X) = (x, y)$. Thus $x = \phi_j(p)$, where $p := \pi(X)$, and $y = \theta_x\big(d\phi_j(X)\big)$. So we have

$$p = \phi_j^{-1}(x), \quad \text{and} \quad X = d\phi_j^{-1}(\theta_x^{-1}(y)).$$

Now note that

$$\begin{aligned}
\theta_{\phi_i} \circ \theta_{\phi_j}^{-1}(x, y) &= \theta_{\phi_i}(X) \\
&= \Big(\phi_i(p), \theta_{\phi_i(p)}\big(d\phi_i(X)\big)\Big) \\
&= \Big(\phi_i \circ \phi_j^{-1}(x), \theta_{\phi_i \circ \phi_j^{-1}(x)}\Big(d(\phi_i \circ \phi_j^{-1})(\theta_x^{-1}(y))\Big)\Big).
\end{aligned}$$

Thus, since $\phi_i \circ \phi_j^{-1}$ is smooth, it follows that $\theta_{\phi_i} \circ \theta_{\phi_j}^{-1}$ is smooth. □

A *vector field* is a mapping $X\colon M \to TM$ such that $\pi(X(p)) = p$ for all $p \in M$. We say that M^n is *parallelizable* if there are n continuos vector fields on M which are linearly independent at each point.

Exercise 15. Show that TM is homeomprphic to $M \times \mathbf{R}^n$ if and only if M is parallelizable. In particular, $T\mathbf{S}^1$ is homeomorphic to $\mathbf{S}^1 \times \mathbf{R}$.

Suppose that to each T_pM there is associated an inner product, i.e., a positive definite symmetric bilinear map $g_p\colon T_pM \times T_pM \to \mathbf{R}$. Then we may define a mapping $f\colon TM \to \mathbf{R}$ by $f(X) := g_{\pi(X)}(X, X)$. If f is smooth, we say that g is a smooth *Riemannian metric*, and (M, g) is a Riemannian manifold. For example, if M is any smooth manifold, and $f\colon M \to \mathbf{R}^n$ is any smooth immersion, then we may define a Riemannian metric on f by

$$g_p(X, Y) := \big\langle df_p(X), df_p(Y)\big\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbf{R}^n$. In particular, since every compact manifold admits a smooth embedding into a Euclidean space, it follows that every compact smooth manifold admits a smooth Riemannian metric. If M is a smooth Riemannian manifold then the *unit tangnet bundle* UTM is defined as the set of tangent vectors of M of length 1.

Exercise 16. Show that the unit tangent bundle of a smooth n-manifold is a smooth $2n - 1$ manifold.

Exercise 17. Show that $T^1\mathbf{S}^2$ is diffeomorphic to $\mathbf{RP}^3$.