Lecture Notes 5

2.2 Definition of Tangent Space

If M is a smooth n-dimensional manifold, then to each point p of M we may associate an n-dimensional vector space T_pM which is defined as follows. Let

$$Curves_pM := \{\, \alpha\colon (-\epsilon, \epsilon) \to M \mid \alpha(0) = p \,\}$$

be the space of *smooth* curves on M centered at p. We say that a pair of curve α, $\beta \in Curves_pM$ are *tangent* at p, and we write $\alpha \sim \beta$, provided that there exists a local chart (U, ϕ) of M centered at p such that

$$(\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0).$$

Note that if (V, ψ) is any other local chart of M centered at p, then, by the chain rule,

$$\begin{aligned}
(\psi \circ \alpha)'(0) &= (\psi \circ \phi^{-1} \circ \phi \circ \alpha)'(0) \\
&= \Big[(\psi \circ \phi^{-1})'(\phi(\alpha(0))\Big] \circ \Big[(\phi \circ \alpha)'(0)\Big] \\
&= \Big[(\psi \circ \phi^{-1})'(\phi(\beta(0))\Big] \circ \Big[(\phi \circ \beta)'(0)\Big] \\
&= (\psi \circ \beta)'(0).
\end{aligned}$$

Thus $\sim$ is well-defined, i.e., it is indpendent of the choice of local coordinates. Further, one may easily check that $\sim$ is an equivalence relation. The set of *tangent vectors* of M at p is defined by

$$T_pM := Curves_pM/\sim.$$

Next we describe how T_pM may be given the structure of a vector space. Let (U, ϕ) denote, as always, a local chart of M centered at p, and recall that $n = \dim(M)$. Then we define a mapping $f\colon T_pM \to \mathbf{R}^n$ by

$$\phi_*([\alpha]) := (\phi \circ \alpha)'(0).$$

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Exercise 1. Show that the above mapping is well-defined and is a bijection.

Since ϕ_* is a bijection, we may use it to identify T_pM with $\mathbf{R}^n$ and, in particular, define a vector space structure on T_pM. More explicitly, we set

$$[\alpha] + [\beta] := \phi_*^{-1}\big(\phi_*([\alpha]) + \phi_*([\beta])\big),$$

and

$$\lambda[\alpha] := \phi_*^{-1}\big(\lambda\phi_*([\alpha])\big).$$

2.3 Derivations

Here we give a more abstract, but useful, characterization for the tangent space of a manifold, which reveals the intimate connection between tangent vectors and directional derivatives.

Let $C^\infty(M)$ denote the space of smooth functions on M and $p \in M$. We say that two functions $f, g \in C^\infty(M)$ have the same *germ* at p, and write $f \sim_p g$, provided that there exists an open neighborhood U of p such that $f|_U = g|_U$. The reslting equivalence classes then defines the space of germ of smooth functions of M at p:

$$C_p(M) := C^\infty M/ \sim_p .$$

Note that we can add and multiply the elements of C_pM in an obvious way, and with respect to these operations one may easily check that $C_p(M)$ is an *algebra* over the field of real numbers $\mathbf{R}$.

We say that a mapping $D\colon C_p(M) \to \mathbf{R}$ is a *derivation* provided that D is linear and satisfies the Leibnitz rule, i.e.,

$$D(fg) = Df \cdot g(p) + f(p) \cdot Dg$$

for all f, $g \in C_p(M)$. If D_1 and D_2 are a pair of such derivations, then we define their sum by $(D_1 + D_2)f := D_1 f + D_2 f$, and for any $\lambda \in \mathbf{R}$, the scalar product is given $(\lambda D)f := \lambda(Df)$.

Exercise 2. Show that the set of derivations of C_pM forms a vector space with respect to the operations defined above.

Note that each element $X \in T_pM$ gives rise to a derivation of $C_p(M)$ if, for any $f \in C_p(M)$, we set

$$Xf := (f \circ \alpha_X)'(0),$$

where $\alpha_X \colon (-\epsilon, \epsilon) \to M$ is a curve which belongs to the equivalence class denoted by X, i.e., $X = [\alpha_X]$.

Exercise 3. Check that Xf is well-defined and is indeed a derivation.

A much less obvious fact, whose demonstrationis the main aim of this section, is that, conversely, every derivation of $C_p(M)$ corresponds to (the directional derivative determined by) a tangent vecor. More formally, if D_pM denotes the space of derivations of C_pM, then

Theorem 4. *T_pM is isomorphic to D_pM.*

The rest of this section is devoted to the proof of the above result. To this end we need a pair of lemmas. Let $\mathbf{0} \in C_p(M)$ denote the constant function zero, i.e. $\mathbf{0}(p) := 0$.

Lemma 5. *If $f \in C_pM$ is a constant function, then $Df = \mathbf{0}$, for any $D \in D_pM$.*

Proof. First note that, since f is constant, say $f(p) = \lambda$,

$$D(f) = D(f \cdot \mathbf{1}) = D(\lambda \cdot \mathbf{1}) = \lambda D(\mathbf{1}),$$

where $\mathbf{1}$ denotes the constant fucntion $\mathbf{1}(p) = 1$. Further,

$$D(\mathbf{1}) = D(\mathbf{1} \cdot \mathbf{1}) = D(\mathbf{1}) \cdot 1 + 1 \cdot D(\mathbf{1}) = 2D(\mathbf{1}).$$

Thus $D(\mathbf{1}) = 0$, which in turn yields that $D(f) = \mathbf{0}$. ☐

Lemma 6. *Let $f \colon \mathbf{R}^n \to \mathbf{R}$ be a smooth function. Then, for any $p \in \mathbf{R}^n$, there exist smooth functions $g^i \colon \mathbf{R}^n \to \mathbf{R}$, $i = 1, \dots, n$, such that*

$$g^i(p) = \frac{\partial f}{\partial x_i}(p),$$

and

$$f(x) = f(p) + \sum_{i=1}^{n} g^i(x)(x^i - p^i).$$

Proof. The fundamental theorem of calculus followed by chain rule implies that

$$\begin{aligned} f(x) - f(p) &= \int_0^1 \frac{d}{dt} f(tp + (1-t)x)dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{dx_i}\Big|_{tp+(1-t)x} (x^i - p^i)dt \\ &= \sum_{i=1}^n \int_0^1 \frac{\partial f}{dx_i}\Big|_{tp+(1-t)x} dt(x^i - p^i). \end{aligned}$$

So we set

$$g^i(x) := \int_0^1 \frac{\partial f}{dx_i}\Big|_{tp+(1-t)x} dt.$$

□

Now we are ready to prove the main result of this section

Proof of Theorem 4. Recall that if (U, ϕ) is a local chart of M centered at p, then the mapping $[\alpha] \mapsto (\phi \circ \alpha)'(0)$ is an isomorphism between T_pM and $\mathbf{R}^n$. Similarly, $f \mapsto f \circ \phi^{-1}$ is an isomorphism between C_pM and $C_o\mathbf{R}^n$, which yields that D_pM is isomorphic to $D_o\mathbf{R}^n$. So it remains to show that $D_o\mathbf{R}^n$ is isomorphic to $\mathbf{R}^n$.

Let $x^i \colon \mathbf{R}^n \to \mathbf{R}$, given by $x^i(p) := p^i$, be the coordinate functions of $\mathbf{R}^n$. It is easy to check that the mapping

$$D_o\mathbf{R}^n \ni D \overset{F}{\longmapsto} (Dx^1, \dots, Dx^n) \in \mathbf{R}^n$$

is a homomorphism. Furhter, F is one-to-one because, by the previous lemmas,

$$Df = 0 + \sum_{i=1}^n (Dg^i \cdot x^i(o) + g^i(o) \cdot Dx^i) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(o) Dx^i.$$

In particular, knowledge of Dx^i uniquely determines D. Finally it remains to show that F is onto. To this end note that to each $X = (X^1, \dots, X^n) \in \mathbf{R}^n$, we may assign a derivation of $C_p\mathbf{R}^n$ given by

$$D_X := \sum_{i=1}^n X^i \frac{\partial}{\partial x_i}\Big|_{x=o}.$$

Then one may quickly check that $F(D_X) = X$. □

Exercise 7. Show that any local chart (U, ϕ) of M centered at p determines a basis $E_1^\phi, \dots E_n^\phi$ for T_pM as follows. For every $f \in C_pM$, set:

$$E_i^\phi f := \frac{\partial (f \circ \phi^{-1})}{\partial x_i}(o).$$

2.4 The differential map

Let $f \colon M \to N$ be a smooth map, and $p \in M$. Then the *differential* of f at p is the mapping $df_p \colon T_pM \to T_{f(p)}N$ given by

$$df_p([\alpha]) := [f \circ \alpha].$$

Exercise 8. Show that if $f \colon \mathbf{R}^n \to \mathbf{R}^m$ and we identify $T_p\mathbf{R}^n$ and $T_{f(p)}\mathbf{R}^m$ with $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively in the standard way (i.e., via the mapping $[\alpha] \mapsto \alpha'(0)$) then df_p may be identified with the linear transformation determined by the jacobian matrix $(\partial f^i / \partial x_j)$ (in particular, df_p is a generalization of the standard derivative $Df(p)$ of maps between Euclidean spaces).

Using the characterization of T_pM as the space of derivations over the germ of smooth functions of M at p, one may give an alternative definition of df_p as follows. Given $X \in T_pM$, we define

$$\big[df_p(X)\big] g := X(g \circ f),$$

for any $g \in C_{f(p)}N$. Thus $df_p(X) \in D_{f(p)}N \simeq T_{f(p)}N$. Note that if $X = [\alpha]$, then

$$X(g \circ f) = (g \circ f \circ \alpha)'(0) = [f \circ \alpha] g.$$

Thus the two definitions of df_p presented above are indeed equivalent. Using the second definition, one may immediately check that df_p is a homomorphism. Another fundamental property is:

Exercise 9 (The chain rule). Show that if $f \colon M \to N$ and $g \colon N \to L$ are smooth maps, then, for any $p \in M$,

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

We say $f \colon M \to N$ is a *diffeomorphism* if f is a homeomorphism, and f and f^{-1} are smooth. If there exists a diffeomorphism between a pair of manifolds we say that these manifolds are diffeomorphic.

Exercise 10. Show that if $f\colon M \to N$ is a diffeomorphism, then df_p is an isomorphism for all $p \in M$. In particular, conclude that if M and N are diffeomorphic, then $dim(M) = dim(N)$.

Note that the last statement if the above exercise also follows from the standard fact in Algebraic topology that $\mathbf{R}^n$ and $\mathbf{R}^m$ are homemorphic only if $m = n$. However, this fact is consequence of homology theory, whereas the above exercise rests only on the basic properties of the differential map. Many results in algebraic toplogy admit more transparent or elegant proofs if one can make use of a differential structure.