Math 598 Jan 21, 2005[1]
Geometry and Topology II
Spring 2005, PSU

Lecture Notes 4

2 Differentiable manifolds

2.1 Differential structures and maps

Recall that a mapping $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is differentiable of class C^r provided that all of its partial derivatives exist and are continuous up to and including order r. If all partial derivatives of f exist up to any order we say that f is C^∞ or *smooth*.

A collection $\{(U_i, \phi_i)\}_{i\in I}$ is called an *atlas* of a manifold M, if U_i cover M and $\phi\colon U_i \to \mathbf{R}^n$ are homeomorphisms. We say that this atlas is C^r if for every i, $j \in I$,

$$\phi_i \circ \phi_j^{-1} \colon \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is C^r. We say that M is a C^r manifold if it admits a C^r atlas.

Exercise 1. Show that $\mathbf{S}^n$ is a smooth (C^∞) manifold.

Exercise 2. Show that $\mathbf{RP}^n$ is a smooth manifold (*Hint*: Define $\mathbf{RP}^n$ as the quotient space $(\mathbf{R}^{n+1} - \{o\})/\sim$, where $x \sim y$ iff $x = \lambda y$ for some $\lambda \neq 0$. Then note that $(\mathbf{R}^{n+1} - \{o\})/\sim$ may be covered by charts U_i consisting of all elements $[x]$ such that the i^{th} coordinate of x is nonzero. In this case the homeomorphisms $\phi_i\colon U_i \to \mathbf{R}^n$ are give by the mappings which omit the i^{th} coordinate x_i and devide the remaining coordinates by x_i.)

Exercise 3. Show that any open subset of a smooth manifold is a smooth manifold.

Exercise 4. Show that the space of $m \times n$ matrices is a smooth manifold, and so is the general linear group $GL(n)$.

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We say that a C^r atlas on M is *maximal* if any local chart which is C^r-*compatible* with the elements of the atlas belongs to the atlas, i.e., if $V \subset M$, and $\psi\colon V \to \mathbf{R}^n$ is a homemorphism such that $\phi_i \circ \psi^{-1}$ and $\psi \circ \phi_i^{-1}$ are C^r then $(V, \psi) \in \{(U_i, \phi_i)\}_{i \in I}$. By a C^r *differential structure* on M we mean the choice of a maximal C^r atlas (in general a manifold may admit several different differential structures, or none at all).

Exercise 5. Show that every C^r atlas on M belongs to a unique differential structure of M.

Thus to specify a differential structure it suffices to specify a C^r atlas. We say a function $f\colon M \to N$ is smooth, if for every $p \in M$ there exist local charts (U, ϕ) of M and (V, ψ) of N, centered at p and $f(p)$ respectively, such that $\psi \circ f \circ \phi^{-1}$ is smooth.

Exercise 6. Show that the notion of smoothness of a function $f\colon M \to N$ is well-defined (i.e., it is independent of the choice of local charts).