CLASSICAL OPEN PROBLEMS IN DIFFERENTIAL GEOMETRY

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By a classical problem in differential geometry I mean one which involves smooth curves or surfaces in three dimensional Euclidean space. We list here a number of such problems. For other problems in differential geometry or geometric analysis see [40]. Some problems and many references may also be found in [6]. A large collection of problems in discrete and convex geometry may be found in [9]. Also see [13] for nice problems invloving convex bodies. For some problems in geometric knot theory see [2].

1. Curves in the plane

The following problem is sometimes called the "bicycle problem", and is equivalent to the 2-dimensional "floating body problem" of Ulam [47, 9]. See [46] for a number of related results, references and questions.

Problem 1 (Chord lenghts of planar curves)**.** *Let* $c\colon \mathbf{R}/L \to \mathbf{R}^2$ *be a closed planar curve of length* L *parametrized by arclength. Suppose that there exists a constant* $s \neq L/2$ *such that* $\|c(t+s) - c(t)\|$ *is constant for all* $t \in \mathbf{R}$*. Does* c *have to be a circle?*

For $s = L/2$, there are non circular curves which satisfy the hypothesis of the above problem [46]. We should also mention that if c is any unit speed closed planar curve, then the average value of the chord lengths $\|c(t+s) - c(t)\|$ reaches its maximum only when c is a circle [1].

Problem 2 (Converse to the four vertex theorem)**.** *Let* $f\colon \mathbf{R}/2\pi \to \mathbf{R}$ *be a continuous periodic function with at least four extremums. Does there exist a closed planar curve* $c\colon \mathbf{R}/2\pi \to \mathbf{R}$ *whose signed curvature at time* t *is* $f(t)$*?*

Herman Gluck has shown that the answer to the above question is yes in the case that $f > 0$ [16].

Problem 3 (Characterizing curvature of closed unit speed curves)**.** *For which periodic functions* $f\colon \mathbf{R}/2\pi \to \mathbf{R}$ *does there exist a unit speed planar curve* $c\colon \mathbf{R}/2\pi \to \mathbf{R}^2$ *whose signed curvature at time* t *is* $f(t)$*?*

The following problem has been solved recently for polygonal curves [8].

Problem 4 (Continuous unfolding)**.** *Let* $c\colon \mathbf{R}/2\pi \to \mathbf{R}^2$ *be a smooth simple closed curve. Show that we can smoothly deform* c *to a convex curve such that the intrinsic*

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distance between any pairs of points of c stays the same while the extrinsic distances in the plane do not get smaller.

2. Curves in Space

The next problem is due to Gromov [17]. Consider a curve in Euclidean space. Then for every pair of points on that curve we have an intrinsic distance and an extrinsic distance. The supremum of the ratio of the intrinsic over the extrinsic distances of distinct pairs of points of the curve is known as the *distortion*. Gromov proved that the closed curve of smallest distortion is the circle [18]. A proof of this may be found in a paper of Kusner and Sullivan [24].

Problem 5 (Universal upper bound for distortion of knot classes)**.** *Does every knot class of closed curves in 3-space has a representative whose distortion is less than some universal constant, say 100?*

A related problem is

Problem 6 (Lower bound for distortion of knots)**.** *What is the smallest distortion that a knot in 3-space can have?*

Recently John Sullivan and Elizabeth Denne have shown that the distortion of a knot has to be at least 3.99 [10].

Problem 7 (Volume of the convex hull of closed curves)**.** *Let* Γ *be a closed curve of fixed length* L *in* $\mathbf{R}^3$*. How big can the volume of the convex hull of* Γ *be.*

For a partial result under symmetry conditions for the above problem see [31]. The above problem has been solved in Euclidean spaces of even dimensions [42]. Also the problem is solved for open arcs in $\mathbf{R}^3$ [11]. For some other related results and questions see [48].

The next problem does not seem to have been considered before.

Problem 8 (Area of the convex hull of closed curves)**.** *Let* Γ *be a closed curve of fixed length* L *in* $\mathbf{R}^3$*, and* A *be the area of the convex hull of* Γ*. Show that* A *is biggest when* Γ *is a circle, in which case we consider the convex hull of* Γ *as a doubly covered disk.*

If Γ is a simple curve which lies on the boundary of its convex hull, then it devides the boundary of the convex hull into a pair of disks each of which has zero curvature. This observation together with the isoperimetric inequality for surfaces of nonpositive curvature, first proved by Andre Weil, may be used to solve the above problem in the case where Γ is simple and lies on the boundary of its convex hull.

The following problem is due to Rosenberg, [38].

Problem 9 (Four vertex theorem for boundary of positively curved surfaces)**.** *Does every curve bounding a surface of positive curvature in* $\mathbf{R}^3$ *has (at least) four points where the torsion vanishes?*

V. D. Sedykh [43] has shown that the answer to the above problem is positive provided that the curve lies on a convex body, as had been conjectured by P. Scherk. Another result which might have some bearing on the above question is a theorem of S. B. Jackson [21] which states that on a simply connected surface of constant positive curvature any closed curve has four extremum points of geodesic curvature.

3. Surfaces

The following problem is due to A. D. Alexandrov [3]:

Problem 10 (Instrisic diameter and area of convex surfaces)**.** *Of all convex surfaces with a fixed intrinsic diameter, is the one with the greatest area a doubled disk?*

It is known that the answer to the above question is affirmative for surfaces of revolution [28] and that in the class of tetrahedra the maximizer is, rather surprisingly, the regular tetrahedron [26]. See also [27] for results relating the intrinsic and extrinsic diameter of convex surfaces. For other partial results using techniques in Riemannian geometry see [45, 39]

The following is in my opinion the outstanding open problem in submanifold geometry:

Problem 11 (Nonflexibility of closed surfaces)**.** *Does there exist a closed smooth surface immersed in Euclidean 3-space which admits a continuous one parameter family of smooth isometric deformations?*

If the surface is convex, then the answer is no; it is known that convex surfaces are rigid. For polyhedral surfaces, the answer to the above question is surprisingly yes! The counterexample was found by Robert Connelly. See [7] for a history of rigidity problems and references.

A closed surface in $\mathbf{R}^3$ is *tight* if its total absolute curvature is as small as possible, or, equivalently, any plane cuts the surface into at most two pieces.

Problem 12 (Rigidity of tight surfaces)**.** *Are there any pairs of smooth tight surfaces which are isometric but not congruent?*

A.D. Alexandrov showed that the answer to the above question is no in the analytic case. See [34] for some partial results in the smooth case. In the polyhedral case Banchoff has shown that the answer in no [4].

It has been shown that a smooth closed curface of positive curvature remains rigid if any points are removed. Can one ontain similar results for other surfacces? In particular,

Problem 13 (Rigidity of the punctured torus)**.** *Does the torus of revolution remain rigid if a point of it is removed?*

The next problem is a conjecture of Meeks:

Problem 14 (Topology of minimal surfaces bounded by convex planar curves)**.** *Is every compact connected minimal surface bounded by a pair of convex planar curves topologically an annulus?*

An affirmative solution to the above problem would lead to a generalization of Shiffman's classical theorems [44] on level set of minimal annuli. Partial or related results have been obtained in [41, 29, 30, 33, 36, 22, 12, 14]. The lecture notes for a talk which I gave at MSRI discusses the history of these results. These lecture notes, together with a video of the lecture, is available on my website.

The next problem is due to John Milnor [32], and would generalize famous theorems of Hilbert and Efimov. Hilbert showed that there exists no complete surfaces of constant negative curvature in $\mathbf{R}^3$, and Effimov proved that there exists no complete surfaces of negative curvature in $\mathbf{R}^3$ whose curvature is bounded away from zero. Proof of Hilbert's theorem may be found in many elementary texts on differential geometry. For Efimov's proof see [32]. Some relatively recent proofs of these results have been announced in [35].

Problem 15 (Principal curvatures of complete minimal surfaces)**.** *Are there any complete surfaces of negative curvature in Euclidean 3-space whose principal curvatures are bounded away from zero?*

Another problem which has been open for a long time is the following conjecture of Caratheordory [23, 19, 25].

Problem 16 (Umbilic points of closed surfaces)**.** *Does every closed surface in Euclidean 3-space have at least two umbilic points.*

The above has been proved only for analytic surfaces, after cumulative efforts by many authors. Another related problem–in fact one which includes the above as a special case–is the following conjecture of Loewner: show that any isolated umbilic point of a surface has index less than 2.

A skew loop is a closed curve without any pairs of parallel tangent lines. It has been shown that the absence of skew loops characterizes the quadric surfaces of positive curvature among all complete surfaces with a point of positive curvature immersed in 3-space [15].

Problem 17 (Skew loops and quadric surfaces)**.** *Does the absence of skew loops characterizes quadric surfaces?*

In other words, remove the assumption that the surface has a point of positive curvature from our theorem.

Total integral of the square of mean curvature is known as Wilmore energy. The following conjecture of Willmore is well known to geometric analysts. See [40] or [6] for references.

Problem 18 (Wilmore energy of tori)**.** *Show that a surface of genus one with the smallest Wilmore energy is a torus of revolution.*

The following problem is also well-known to people who work on mean curvature:

Problem 19 (Constant mean curvature surfaces bounded by a circle)**.** *Does there exist a compact surface of constant mean curvature which is bounded by a circle, but is not a piece of a sphere.*

A related question has been posed by Ros and Rosenberg [37]:

Problem 20 (Constant mean curvature surfaces bounded by convex curves)**.** *Show that any compact embedded CMC surface bounded by a convex planar curve is topologically a disk.*

The above problem has been affirmatively solved by Barbosa and Jorge assuming that the surface is stable [5].

Problem 21 (Volume of surfaces of constant width)**.** *Let $S \subset \mathbf{R}^3$ be a closed surface of constant width and fixed area. How small can the volume of S be?*

The above problem has been solved in $\mathbf{R}^2$: the Reuleux triangle has the least area among all closed curves of constant width. For a recent proof of this result, which is originally due to Blaschke and Lebegue, see [20]. For other references see [9].

Problem 22 (Surfaces with strips of constant area)**.** *Let $S \subset \mathbf{R}^3$ be a closed surface of diameter d. Suppose that there exists a constant $h < d$ so that whenever a pair of planes separated by a distance of h intersect S, the area of S contained between these planes is constant. Does it then follow that S is a sphere?*

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