Nov 11, 2004[1]

Lecture Notes 13

2.9 The Covariant Derivative, Lie Bracket, and Riemann Curvature Tensor of $\mathbf{R}^n$

Let $A \subset \mathbf{R}^n$, $p \in A$, and W be a *tangent vector* of A at p, i.e., suppose there exists a curve $\gamma\colon (-\epsilon, \epsilon) \to A$ with $\gamma(0) = p$ and $\gamma'(0) = W$. Then if $f\colon A \to \mathbf{R}$ is a function we define the (directional) derivative of f with respect to W at p as

$$W_p f := (f \circ \gamma)'(0) = df_p(W).$$

Similarly, if V is a *vectorfield* along A, i.e., a mapping $V\colon A \to \mathbf{R}^n$, $p \mapsto V_p$, we define the *covariant derivative* of V with respect to W at p as

$$\overline{\nabla}_{W_p} V := (V \circ \gamma)'(0) = dV_p(W).$$

Note that if f and V are C^1, then by definition they may be extended to an open neighborhood of A. So df_p and dV_p, and consequently $W_p f$ and $\overline{\nabla}_{W_p} V$ are well defined. In particular, they do not depend on the choice of the curve γ or the extensions of f and V.

Exercise 1. Let E_i be the standard basis of $\mathbf{R}^n$, i.e., $E_1 := (1, 0, \dots, 0)$, $E_2 := (0, 1, 0, \dots, 0), \dots, E_n := (0, \dots, 0, 1)$. Show that for any functions $f\colon \mathbf{R}^n \to \mathbf{R}$ and vectorfield $V\colon \mathbf{R}^n \to \mathbf{R}^n$

$$(E_i)_p f = D_i f(p) \quad \text{and} \quad \overline{\nabla}_{(E_i)_p} V = D_i V(p)$$

(*Hint:* Let $u_i\colon (-\epsilon, \epsilon) \to \mathbf{R}^n$ be given by $u_i(t) := p + tE_i$, and observe that $(E_i)_p f = (f \circ u_i)'(0)$, $\overline{\nabla}_{(E_i)_p} V = (V \circ u_i)'(0)$).

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The operation $\overline{\nabla}$ is also known as the standard *Levi-Civita* connection of $\mathbf{R}^n$. If W is a tangent vectorfield of A, i.e., a mapping $W\colon A \to \mathbf{R}^n$ such that W_p is a tangent vector of A for all $p \in A$, then we set

$$Wf(p) := W_p f \quad \text{and} \quad (\overline{\nabla}_W V)_p := \overline{\nabla}_{W_p} V.$$

Note that $Wf\colon A \to \mathbf{R}$ is a function and $\overline{\nabla}_W V$ is a vectorfield. Further, we define

$$(fW)_p := f(p)W_p.$$

Thus $fW\colon A \to \mathbf{R}^n$ is a also a vector field.

Exercise 2. Show that it $V = (V^1, \dots, V^n)$, i.e., V^i are the component functions of V, then

$$\overline{\nabla}_W V = (WV^1, \dots, WV^n).$$

Exercise 3. Show that if Z is a tangent vectorfield of A and $f\colon A \to \mathbf{R}$ is a function, then

$$\overline{\nabla}_{W+Z} V = \overline{\nabla}_W V + \overline{\nabla}_Z V, \quad \text{and} \quad \overline{\nabla}_{fW} V = f\overline{\nabla}_W V.$$

Further if $Z\colon A \to \mathbf{R}^n$ is any vectorfield, then

$$\overline{\nabla}_W (V + Z) = \overline{\nabla}_W V + \overline{\nabla}_W Z, \quad \text{and} \quad \overline{\nabla}_W (fV) = (Wf)V + f\overline{\nabla}_W V.$$

Exercise 4. Note that if V and W are a pair of vectorfields on A then $\langle V, W\rangle\colon A \to \mathbf{R}$ defined by $\langle V, W\rangle_p := \langle V_p, W_p\rangle$ is a function on A, and show that

$$Z\langle V, W\rangle = \left\langle \overline{\nabla}_Z V, W\right\rangle + \left\langle V, \overline{\nabla}_Z W\right\rangle.$$

If $V, W\colon A \to \mathbf{R}^n$ are a pair of vector fields, then their *Lie bracket* is the vector filed on A defined by

$$[V, W]_p := \overline{\nabla}_{V_p} W - \overline{\nabla}_{W_p} V.$$

Exercise 5. Show that if $A \subset \mathbf{R}^n$ is open, $V, W\colon A \to \mathbf{R}^n$ are a pair of vector fields and $f\colon A \to \mathbf{R}$ is a scalar, then

$$[V, W]f = V(Wf) - W(Vf).$$

(*Hint:* First show that $Vf = \langle V, \operatorname{grad} f\rangle$ and $Wf = \langle W, \operatorname{grad} f\rangle$ where

$$\operatorname{grad} f := (D_1 f, \dots, D_n f).$$

Next define

$$\operatorname{Hess} f(V, W) := \langle V, \nabla_W \operatorname{grad} f\rangle,$$

and show that $\operatorname{Hess} f(V, W) = \operatorname{Hess} f(W, V)$. In particular, it is enough to show that $\operatorname{Hess} f(E_i, E_j) = D_{ij} f$, where $\{E_1, \dots, E_n\}$ is the standard basis for $\mathbf{R}^n$. Then Leibnitz rule yields that

$$\begin{gathered} V(Wf) - W(Vf) \\ = V\langle W, \operatorname{grad} f\rangle - W\langle V, \operatorname{grad} f\rangle \\ = \langle \nabla_V W, \operatorname{grad} f\rangle + \langle W, \nabla_V \operatorname{grad} f\rangle - \langle \nabla_W V, \operatorname{grad} f\rangle - \langle V, \nabla_W \operatorname{grad} f\rangle \\ = \langle [V, W], \operatorname{grad} f\rangle + \operatorname{Hess} f(W, V) - \operatorname{Hess} f(V, W) \\ = [V, W] f, \end{gathered}$$

as desired.)

If V and W are tangent vectorfields on an open set $A \subset \mathbf{R}^n$, and $Z\colon A \to \mathbf{R}^n$ is any vectorfield, then

$$\overline{R}(V, W)Z := \overline{\nabla}_V \overline{\nabla}_W Z - \overline{\nabla}_W \overline{\nabla}_V Z - \overline{\nabla}_{[V,W]} Z$$

defines a vectorfield on A. If Y is another vectorfield on A, then we may also define an associated scalar quantity by

$$\overline{R}(V, W, Z, Y) := \left\langle \overline{R}(V, W)Z, Y \right\rangle,$$

which is known as the *Riemann curvature tensor* of $\mathbf{R}^n$.

Exercise 6. Show that $\overline{R} \equiv 0$.

2.10 The Induced Covariant Derivative on Surfaces; Gauss's Formulas revisited

Note that if $M \subset \mathbf{R}^3$ is a regular embedded surface and V, $W\colon M \to \mathbf{R}^3$ are vectorfields on M. Then $\overline{\nabla}_W V$ may no longer be tangent to M. Rather, in general we have

$$\overline{\nabla}_W V = \left(\overline{\nabla}_W V\right)^{\top} + \left(\overline{\nabla}_W V\right)^{\perp},$$

where $(\overline{\nabla}_W V)^\top$ and $(\overline{\nabla}_W V)^\perp$ respectively denote the tangential and normal components of $\overline{\nabla}_W V$ with resect to M. More explicitly, if for each $p \in M$ we let $n(p)$ be a unit normal vector to T_pM, then

$$\big(\overline{\nabla}_W V\big)_p^\perp := \big\langle \overline{\nabla}_{W_p} V, n(p)\big\rangle n(p) \quad \text{and} \quad \big(\overline{\nabla}_W V\big)^\top := \overline{\nabla}_W V - \big(\overline{\nabla}_W V\big)^\perp.$$

Let $\mathcal{X}(M)$ denote the space of tangent vectorfield on M. Then We define the *(induced) covariant derivative* on M as the mapping $\nabla\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$\nabla_W V := \big(\overline{\nabla}_W V\big)^\top.$$

Exercise 7. Show that, with respect to tangent vectorfields on M, ∇ satisfies all the properties which were listed for $\overline{\nabla}$ in Exercises 3 and 4.

Next we derive an explicit expression for ∇ in terms of local coordinates. Let $X\colon U \to M$ be a proper regular patch centered at a point $p \in M$, i.e., $X(0,0) = p$, and set

$$\overline{X}_i := X_i \circ X^{-1}.$$

Then $\overline{X}_i$ are vectorfields on $X(U)$, and for each $q \in X(U)$, $(\overline{X}_i)_q$ forms a basis for T_qM. Thus on $X(U)$ we have

$$V = \sum_i V^i \overline{X}_i, \quad \text{and} \quad W = \sum_i W^i \overline{X}_i$$

for some functions V^i, $W^i\colon X(U) \to \mathbf{R}$. Consequently, on $X(U)$,

$$\begin{aligned}
\nabla_W V &= \nabla_{(\sum_j W^j \overline{X}_j)}\Big(\sum_i V^i \overline{X}_i\Big) \\
&= \sum_j \left(W^j \nabla_{\overline{X}_j}\Big(\sum_i V^i \overline{X}_i\Big)\right) \\
&= \sum_j \left(W^j \sum_i \Big(\overline{X}_j V^i + V^i \nabla_{\overline{X}_j} \overline{X}_i\Big)\right) \\
&= \sum_j \sum_i \Big(W^j \big(\overline{X}_j V^i\big) + W^j V^i \nabla_{\overline{X}_j} \overline{X}_i\Big).
\end{aligned}$$

Next note that if we define $u_j\colon (-\epsilon, \epsilon) \to \mathbf{R}^2$ by $u_j(t) := tE_j$, where $E_1 := (1,0)$ and $E_2 := (0,1)$. Then $X \circ u_i\colon (-\epsilon, \epsilon) \to M$ are curves with $X \circ u_i(0) =$

p and $(X \circ u_i)'(0) = X_i(0,0) = \overline{X}_i(p)$. Thus by the definitions of ∇ and $\overline{\nabla}$ we have

$$\begin{aligned}
\nabla_{(\overline{X}_j)_p} \overline{X}_i &= \Big(\overline{\nabla}_{(\overline{X}_j)_p} \overline{X}_i\Big)^\top \\
&= \Big(\big(\overline{X}_i \circ (X \circ u_j)\big)'(0)\Big)^\top \\
&= \Big((X_i \circ u_j)'(0)\Big)^\top
\end{aligned}$$

Now note that, by the chain rule,

$$(X_i \circ u_j)'(0) = DX_i\big(u_j(0)\big)Du_j(0) = X_{ij}(0,0).$$

Exercise 8. Verify the last equality above.

Thus, by Gauss's formula,

$$\begin{aligned}
\nabla_{(\overline{X}_j)_p} \overline{X}_i &= \Big(X_{ij}(0,0)\Big)^\top \\
&= \Big(\sum_k \Gamma_{ij}^k(0,0) X_k(0,0) + l_{ij}(0,0) N(0,0)\Big)^\top \\
&= \sum_k \Gamma_{ij}^k\big(X^{-1}(p)\big) X_k\big(X^{-1}(p)\big) \\
&= \sum_k \Gamma_{ij}^k\big(X^{-1}(p)\big) \big(\overline{X}_k\big)_p.
\end{aligned}$$

In particular if we set $\overline{X}_{ij} := X_{ij} \circ X^{-1}$ and define $\overline{\Gamma}_{ij}^k \colon X(U) \to \mathbf{R}$ by $\overline{\Gamma}_{ij}^k := \Gamma_{ij}^k \circ X^{-1}$, then we have

$$\nabla_{\overline{X}_j} \overline{X}_i = \big(\overline{X}_{ij}\big)^\top = \sum_k \overline{\Gamma}_{ij}^k \overline{X}_k,$$

which in turn yields

$$\nabla_W V = \sum_j \sum_i \Big(W^j \overline{X}_j V^i + W^j V^i \sum_k \overline{\Gamma}_{ij}^k \overline{X}_k\Big).$$

Now recall that Γ_{ij}^k depends only on the coefficients of the first fundamental form g_{ij}. Thus it follows that ∇ is intrinsic:

Exercise 9. Show that if $f\colon M \to \widetilde{M}$ is an isometry, then

$$\widetilde{\nabla}_{df(W)} df(V) = df\big(\nabla_W V\big),$$

where $\widetilde{\nabla}$ denotes the covariant derivative on $\widetilde{M}$ (*Hint:* It is enough to show that $\langle \widetilde{\nabla}_{df(\overline{X}_i)} df(\overline{X}_j), df(\overline{X}_l)\rangle = \langle df\big(\nabla_{\overline{X}_i}\overline{X}_j\big), df(\overline{X}_l)\rangle$).

Next note that if $n\colon X(U) \to \mathbf{S}^2$ is a local Gauss map then

$$\langle \nabla_W V, n\rangle = -\langle V, \nabla_W n\rangle = -\langle V, dn(W)\rangle = \langle V, S(W)\rangle,$$

where, recall that, S is the shape operator of M. Thus

$$\big(\overline{\nabla}_{W_p} V\big)^{\perp} = \langle V, S(W_p)\rangle n(p),$$

which in turn yields

$$\overline{\nabla}_W V = \nabla_W V + \big\langle V, S(W)\big\rangle n.$$

This is Gauss's formula and implies the expression that we had derived earlier in local coordinates.

Exercise 10. Verify the last sentence.