

Lecture Notes 11

2.4 Intrinsic Metric and Isometries of Surfaces

Let $M \subset \mathbf{R}^3$ be a regular embedded surface and $p, q \in M$, then we define

$$\text{dist}_M(p, q) := \inf\{\text{Length}[\gamma] \mid \gamma: [0, 1] \rightarrow M, \gamma(0) = p, \gamma(1) = q\}.$$

Exercise 1. Show that (M, dist_M) is a metric space.

Lemma 2. Show that if M is a C^1 surface, and $X \subset M$ is compact, then for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\text{dist}_M(p, q) - \|p - q\|| \leq \epsilon \|p - q\|$$

for all $p, q \in X$, with $\text{dist}_M(p, q) \leq \delta$.

Proof. Define $F: M \times M \rightarrow \mathbf{R}$ by $F(p, q) := \text{dist}_M(p, q)/\|p - q\|$, if $p \neq q$ and $F(p, q) := 1$ otherwise. We claim that F is continuous. To see this let p_i be a sequence of points of M which converge to a point $p \in M$. We may assume that p_i are contained in a Monge patch of M centered at p given by

$$X(u_1, u_2) = (u_1, u_2, h(u_1, u_2)).$$

Let x_i and y_i be the x and y coordinates of p_i . If p_i is sufficiently close to $p = (0, 0)$, then, since $\nabla h(0, 0) = 0$, we can make sure that

$$\|\nabla h(tx_i, ty_i)\|^2 \leq \epsilon,$$

for all $t \in [0, 1]$ and $\epsilon > 0$. Let $\gamma: [0, 1] \rightarrow \mathbf{R}^3$ be the curve given by

$$\gamma(t) = (tx_i, ty_i, h(tx_i, ty_i)).$$

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Then, since γ is a curve on M ,

$$\begin{aligned}
\text{dist}_M(p, p_i) &\leq \text{Length}[\gamma] \\
&= \int_0^1 \sqrt{x_i^2 + y_i^2 + \langle \nabla h(tx_i, ty_i), (x_i, y_i) \rangle^2} dt \\
&\leq \int_0^1 \sqrt{x_i^2 + y_i^2 + \epsilon(x_i^2 + y_i^2)^2} dt \\
&\leq \sqrt{1 + \epsilon} \sqrt{x_i^2 + y_i^2} \\
&\leq (1 + \epsilon) \|p - p_i\|
\end{aligned}$$

So, for any $\epsilon > 0$ we have

$$1 \leq \frac{\text{dist}_M(p, p_i)}{\|p - p_i\|} \leq 1 + \epsilon$$

provided that p_i is sufficiently close to p . We conclude then that F is continuous. So $U := F^{-1}([1, 1 + \epsilon])$ is an open subset of $M \times M$ which contains the diagonal $\Delta_M := \{(p, p) \mid p \in M\}$. Since $\Delta_X \subset \Delta_M$ is compact, we may then choose δ so small that $V_\delta(\Delta_X) \subset U$, where $V_\delta(\Delta_X)$ denotes the open neighborhood of Δ_X in $M \times M$ which consists of all pairs of points (p, q) with $\text{dist}_M(p, q) \leq \delta$. \square

Exercise 3. Does the above lemma hold also for C^0 surfaces?

If $\gamma: [a, b] \rightarrow M$ is any curve then we may define

$$\begin{aligned}
\text{Length}_M[\gamma] &:= \\
&\sup \left\{ \sum_{i=1}^k \text{dist}_M(\gamma(t_i), \gamma(t_{i-1})) \mid \{t_0, \dots, t_k\} \in \text{Partition}[a, b] \right\}.
\end{aligned}$$

Lemma 4. $\text{Length}_M[\gamma] = \text{Length}[\gamma]$.

Proof. Note that

$$\text{dist}_M(\gamma(t_i), \gamma(t_{i-1})) \geq \|\gamma(t_i) - \gamma(t_{i-1})\|.$$

Thus $\text{Length}_M[\gamma] \geq \text{Length}[\gamma]$. Further, by the previous lemma, we can make sure that

$$\text{dist}_M(\gamma(t_i), \gamma(t_{i-1})) \leq (1 + \epsilon) \|\gamma(t_i) - \gamma(t_{i-1})\|,$$

which yields $\text{Length}_M[\gamma] \leq (1 + \epsilon) \text{Length}[\gamma]$, for any $\epsilon > 0$. \square

We say that $f: M \rightarrow \overline{M}$ is an *isometry* provided that

$$\text{dist}_{\overline{M}}(f(p), f(q)) = \text{dist}_M(p, q).$$

Lemma 5. $f: M \rightarrow \overline{M}$ is an isometry, if and only if $\text{Length}[\gamma] = \text{Length}[f \circ \gamma]$ for all curves $\gamma: [a, b] \rightarrow M$.

Proof. If f is an isometry, then, by the previous lemma,

$$\text{Length}[\gamma] = \text{Length}_M[\gamma] = \text{Length}_{\overline{M}}[f \circ \gamma] = \text{Length}_M[f \circ \gamma].$$

The converse is clear. □

Exercise 6. Justify the middle equality in the last expression displayed above.

Theorem 7. $f: M \rightarrow \overline{M}$ is an isometry if and only if for all $p \in M$, and $v, w \in T_pM$,

$$\langle df_p(v), df_p(w) \rangle = \langle v, w \rangle.$$

Proof. Suppose that f is an isometry. Let $\gamma: (-\epsilon, \epsilon) \rightarrow M$ be a curve with $\gamma(0) = p$, and $\gamma'(0) = v$. Then, by the previous lemma

$$\int_{-\epsilon}^{\epsilon} \|\gamma'(t)\| dt = \int_{-\epsilon}^{\epsilon} \|(f \circ \gamma)'(t)\| dt$$

Taking the limit of both sides as $\epsilon \rightarrow 0$ and applying the mean value theorem for integrals, yields then that

$$\|v\| = \|\gamma'(0)\| = \|(f \circ \gamma)'(0)\| = \|df_p(v)\|.$$

Thus df preserves the norm, which implies that it must preserve the inner-product as well (see the following exercise).

Conversely, suppose that $\|v\| = \|df_p(v)\|$. Then, if $\gamma: [a, b] \rightarrow M$ is any curve, we have

$$\int_a^b \|(f \circ \gamma)'(t)\| dt = \int_a^b \|df_{\gamma(t)}(\gamma'(t))\| dt = \int_a^b \|\gamma'(t)\| dt.$$

So f preserves the length of all curves, which, by the previous Lemma, shows that f is an isometry. □

Exercise 8. Show that a function $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ preserves the norm $\|\cdot\|$ if and only if it preserves the inner product $\langle \cdot, \cdot \rangle$.

2.5 Gauss's Theorema Egregium

Lemma 9. *Let $X: U \rightarrow M$ be a proper regular chart. Then $\bar{X} := f \circ X: U \rightarrow \bar{M}$ is a proper regular chart as well and $g_{ij} = \bar{g}_{ij}$ on U .*

Proof. Using the last theorem we have

$$\begin{aligned}
 \bar{g}_{ij}(u_1, u_2) &= \langle D_i \bar{X}(u_1, u_2), D_j \bar{X}(u_1, u_2) \rangle \\
 &= \langle D_i(f \circ X)(u_1, u_2), D_j(f \circ X)(u_1, u_2) \rangle \\
 &= \langle df_{X(u_1, u_2)}(D_i X(u_1, u_2)), df_{X(u_1, u_2)}(D_j X(u_1, u_2)) \rangle \\
 &= \langle D_i X(u_1, u_2), D_j X(u_1, u_2) \rangle \\
 &= g_{ij}(u_1, u_2).
 \end{aligned}$$

□

Exercise 10. Justify the third equality in the last displayed expression above.

Let \mathcal{F} denote the set of functions $f: U \rightarrow \mathbf{R}$ where $U \subset \mathbf{R}^2$ is an open neighborhood of the origin.

Lemma 11. *There exists a mapping Briochi: $\mathcal{F} \times \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ such that for any chart $X: U \rightarrow M$ centered at $p \in M$,*

$$K(p) = \text{Briochi}[g_{11}, g_{12}, g_{22}](0, 0).$$

Proof. Recall that

$$K(p) = \frac{\det l_{ij}(0, 0)}{\det g_{ij}(0, 0)},$$

and, by Lagrange's identity,

$$l_{ij} = \left\langle X_{ij}, \frac{X_1 \times X_2}{\|X_1 \times X_2\|} \right\rangle = \frac{1}{\sqrt{\det g_{ij}}} \langle X_{ij}, X_1 \times X_2 \rangle,$$

where $X_{ij} := D_{ij}X$, and $X_i := D_i X$. Thus

$$K(p) = \frac{\det(\langle X_{ij}(0, 0), X_1(0, 0) \times X_2(0, 0) \rangle)}{(\det g_{ij}(0, 0))^2}.$$

Next note that

$$\det(\langle X_{ij}, X_1 \times X_2 \rangle) = \langle X_{11}, X_1 \times X_2 \rangle \langle X_{21}, X_1 \times X_2 \rangle - \langle X_{12}, X_1 \times X_2 \rangle^2$$

The right hand side of the last expression may be rewritten as

$$\det(X_{11}, X_1, X_2) \det(X_{22}, X_1, X_2) - (\det(X_{12}, X_1, X_2))^2,$$

where (u, v, w) here denotes the matrix with columns u , v , and w . Recall that if A is a square matrix with transpose A^T , then $\det A = \det A^T$. Thus the last expression displayed above is equivalent to

$$\det((X_{11}, X_1, X_2)^T (X_{22}, X_1, X_2)) - \det((X_{12}, X_1, X_2)^T (X_{12}, X_1, X_2)),$$

which in turn can be written as

$$\det \begin{pmatrix} \langle X_{11}, X_{22} \rangle & \langle X_{11}, X_1 \rangle & \langle X_{11}, X_2 \rangle \\ \langle X_1, X_{22} \rangle & \langle X_1, X_1 \rangle & \langle X_1, X_2 \rangle \\ \langle X_2, X_{22} \rangle & \langle X_2, X_1 \rangle & \langle X_2, X_2 \rangle \end{pmatrix} - \det \begin{pmatrix} \langle X_{12}, X_{12} \rangle & \langle X_{12}, X_1 \rangle & \langle X_{12}, X_2 \rangle \\ \langle X_1, X_{12} \rangle & \langle X_1, X_1 \rangle & \langle X_1, X_2 \rangle \\ \langle X_2, X_{12} \rangle & \langle X_2, X_1 \rangle & \langle X_2, X_2 \rangle \end{pmatrix}.$$

If we expand the above determinants along their first rows, then $\langle X_{11}, X_{22} \rangle$ and $\langle X_{12}, X_{22} \rangle$ will have the same coefficients. This implies that we can rewrite the last expression as

$$\det \begin{pmatrix} \langle X_{11}, X_{22} \rangle - \langle X_{12}, X_{12} \rangle & \langle X_{11}, X_1 \rangle & \langle X_{11}, X_2 \rangle \\ \langle X_1, X_{22} \rangle & \langle X_1, X_1 \rangle & \langle X_1, X_2 \rangle \\ \langle X_2, X_{22} \rangle & \langle X_2, X_1 \rangle & \langle X_2, X_2 \rangle \end{pmatrix} - \det \begin{pmatrix} 0 & \langle X_{12}, X_1 \rangle & \langle X_{12}, X_2 \rangle \\ \langle X_1, X_{12} \rangle & \langle X_1, X_1 \rangle & \langle X_1, X_2 \rangle \\ \langle X_2, X_{12} \rangle & \langle X_2, X_1 \rangle & \langle X_2, X_2 \rangle \end{pmatrix}.$$

Now note that each of the entries in the above matrices can be expressed purely in terms of g_{ij} , since

$$\begin{aligned} \langle X_{ii}, X_j \rangle &= \frac{1}{2} \langle X_i, X_i \rangle_j = \frac{1}{2} (g_{ii})_j, \\ \langle X_{ij}, X_i \rangle &= \langle X_i, X_j \rangle_i - \langle X_i, X_{ji} \rangle = (g_{ij})_i - \frac{1}{2} (g_{ii})_j, \end{aligned}$$

and

$$\begin{aligned} \langle X_{11}, X_{22} \rangle - \langle X_{12}, X_{12} \rangle &= \langle X_1, X_{22} \rangle_1 - \langle X_1, X_{12} \rangle_2 \\ &= (g_{21})_{21} - \frac{1}{2} (g_{11})_{21} - \frac{1}{2} (g_{11})_2. \end{aligned}$$

Substituting the above values in the previous matrices, we define

$$\begin{aligned} \text{Briochi}[g_{11}, g_{22}, g_{33}] := & \\ \frac{1}{(\det(g_{ij}))^2} & \left(\det \begin{pmatrix} (g_{21})_{21} - \frac{1}{2}(g_{11})_{21} - \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{11})_1 & \frac{1}{2}(g_{11})_2 \\ (g_{21})_2 - \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_2 & g_{21} & g_{22} \end{pmatrix} \right. \\ & \left. - \det \begin{pmatrix} 0 & \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{22})_1 \\ \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_1 & g_{21} & g_{22} \end{pmatrix} \right). \end{aligned}$$

Evaluating the above expression at $(0,0)$ yields that Gaussian curvature $K(p)$. \square

Theorem 12. *If $f: M \rightarrow \overline{M}$ is an isometry, then $\overline{K}(f(p)) = K(p)$, where K and \overline{K} denote the Gaussian curvatures of M and \overline{M} respectively.*

Proof. Let $X: U \rightarrow M$ be a chart centered at p , then $\overline{X} := f \circ X$ is a chart of \overline{M} centered at $f(p)$. Let g_{ij} and \overline{g}_{ij} denote the coefficients of the first fundamental form with respect to the charts X and \overline{X} respectively. Then, using the previous two lemmas, we have

$$\begin{aligned} \overline{K}(f(p)) &= \text{Briochi}[\overline{g}_{11}, \overline{g}_{12}, \overline{g}_{22}](0,0) \\ &= \text{Briochi}[g_{11}, g_{12}, g_{22}](0,0) \\ &= K(p). \end{aligned}$$

\square

Exercise 13. Let $M \subset \mathbf{R}^3$ be a regular embedded surface and $p \in M$. Suppose that $K(p) \neq 0$. Does there exist a chart $X: U \rightarrow M$ such that D_1X and D_2X are orthonormal at all points of U .