Math 426 Feb 9, 2004[1]
Introduction to Modern Geometry
Spring 2004, PSU

Lecture Notes 5

1.11 The four vertex theorem

A *vertex* of a planar curve $\alpha\colon I \to \mathbf{R}^2$ is a point where the curvature of α has a local max or min.

Exercise 1. Show that an ellipse has exactly 4 vertices, unless it is a circle.

The main aim of this section is to show that:

Theorem 2. *Any convex planar curve has (at least) four vertices.*

In fact any simple closed curve has 4 vertices, but the proof is hader. On the other hand if the curve is not simple, then the 4 vertex property may no longer be true:

Exercise 3. Sketch the limacon $\alpha\colon [0, 2\pi] \to \mathbf{R}^2$ given by

$$\alpha(t) := (2\cos t + 1)(\cos t, \sin t)$$

and show that it has only two vertices. (*Hint*: It looks like a loop with a smaller loops inside)

The proof of the above theorem is by contradiction. Suppose that α has fewer than 4 vertices, then it must have exactly 2.

Exercise 4. Verify the last sentence.

Suppose that these two vertices occur at t_0 and t_1. Then $\kappa'(t)$ will have one sign on (t_1, t_2) and the opposite sign on $I - [t_1, t_2]$. Let ℓ be the line passing through $\alpha(t_1)$ and $\alpha(t_2)$. Then, since α is convex, α restricted to (t_1, t_2) lies on side of ℓ and α restricted to $I - [t_1, t_2]$ lies on the other side. Let p be a point of ℓ and v be a vector orthogonal to ℓ, then $f\colon I \to \mathbf{R}$, given

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by $f(t) := \langle \alpha(t) - p, v\rangle$ has one sign on (t_1, t_2) and has the opposite sign on $I - [t_1, t_2]$. Consequently, $\kappa'(t) f(t)$ is always nonnegative. So

$$0 < \int_I \kappa'(t)\langle \alpha(t) - p, v\rangle dt.$$

On the other hand

$$\begin{aligned}
\int_I \kappa'(t)\langle \alpha(t) - p, v\rangle dt &= \kappa(t)\langle \alpha(t) - p, v\rangle|_a^b - \int_I \kappa(t)\langle T(t) - p, v\rangle dt \\
&= 0 - \int_I \langle -N'(t) - p, v\rangle dt \\
&= \langle -N(t) - p, v\rangle|_a^b \\
&= 0.
\end{aligned}$$

So we have a contradiction, as desired.

Exercise 5. Justify each of the lines in the above computation.

1.12 Isoperimetric inequality

Let $\alpha\colon I \to \mathbf{R}^2$ be a simple closed planar curve. By the Jordan curve theorem (which we will not prove here), $\mathbf{R}^2 - \alpha(I)$ consists of two connected components, and the boundary of each component is $\alpha(I)$. Further, one of these components is contained in some large sphere, while the other is unbounded. The two dimensional Lebegue measure of the bounded component is what we call the area of α.

Theorem 6. *For any simple closed planar curve $\alpha\colon I \to \mathbf{R}^2$,*

$$Area[\alpha] \leq \frac{Length[\alpha]^2}{4\pi}.$$

Equality holds only when α is a circle.

Our proof of the above theorem hinges on the following subtle fact whose proof we leave out

Lemma 7. *Of all simple closed curves of fixed length L, there exists at least one with the biggest area. Further, every such curve is C^1.*

Exercise* 8. Show that the area maximizer (for a fixed length) must be convex. (*Hint*: It is enough to show that if the maximizer, say α, is not convex, then there exist a line ℓ with respect to which $\alpha(I)$ lies on one side, and intersects $\alpha(I)$ at two points p and q but not in the intervening open segment of ℓ determined by p and q. Then reflecting one of the segments of $\alpha(I)$, determined by p and q, through ℓ increases area while leaving the length unchanged.)

We say that α is symmetric with respect to a line ℓ provided that the image of α is invariant under reflection with respect to ℓ.

Exercise 9. Show that a C^1 convex planar curve $\alpha\colon I \to \mathbf{R}^2$ is a circle, if and only if for every unit vector $u \in \mathbf{S}^1$ there exists a line perpendicular to u with respect to which α is symmetric (*Hint* Suppose that α has a line of symmetry in every direction. First show that each line of symmetry is unique in the corresponding direction. After a translation we may assume that α is symmetric with respect to both the x-axis and the y-axis. Show that this yields that α is symmetric with respect to the origin, i.e. rotation by 180°. From this and the uniqueness of the lines of symmetry conclude that every line of symmetry passes through the origin. Finally show that each line of symmetry must meet the curve orthogonally at the intersection points. This shows that $\langle \alpha(t), \alpha'(t) \rangle = 0$, which in turn yields that $\|\alpha(t)\| = const.$)

Let $\alpha\colon I \to \mathbf{R}^2$ be an area maximizer. By Exercise 8 we may assume that α is convex. We claim that α must have a line of symmetry in every direction, which would show, by Exercise 9, that α is a circle, and hence would complete the proof.

Suppose, towards a contradiction, that there exists a direction $u \in \mathbf{S}^1$ such that α has no line of symmetry in that direction. Ater a rigid motion, we may assume that $u = (0, 1)$.

Let $[a, b]$ be the projection of $\alpha(I)$ to the x-axis. Then, since α is convex, every vertical line which passes through an interior point of (a, b) intersects $\alpha(I)$ at precisely two points. Let $f(x)$ be the y-coordinate of the higher point, and $g(x)$ be the y-coordinate of the other points. Then

$$Area[\alpha] = \int_a^b f(x) - g(x)\, dx.$$

Further note that if α is C^1 then f and g are C^1 as well, thus

$$Length[\alpha] = f(a) - g(a) + \int_a^b \sqrt{1 + f'(x)^2}\, dx + \int_a^b \sqrt{1 + g'(x)^2}\, dx + f(b) - g(b).$$

Now we are going to define a new curve $\overline{\alpha}$ which is bounded above by the graph of the function $\overline{f}\colon [a,b] \to \mathbf{R}$ given by

$$\overline{f}(x) := \frac{f(x) - g(x)}{2},$$

is bounded below by the graph of $-\overline{f}$, and is bouned on the left and right by vertical segments, which may consist only of a single point. One immdediately checks that

$$Area[\overline{\alpha}] = Area[\alpha].$$

Further, note that since by assumption α is not symmetric with respect to the x-axis, $\overline{f}$ is strictly positive on (a,b). This may be used to show that

$$Length[\overline{\alpha}] < Length[\alpha].$$

Exercise 10. Verify the last inequlaity above (*Hint*: It is enough to check that $\int_a^b \sqrt{1 + \overline{f}'(x)^2}dx$ is strictly smaller that either of the integrals in the above formula for the length of α).