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Introduction to Modern Geometry
Spring 2004, PSU

Lecture Notes 3

1.4 Curves of Constant Curvature

Here we show that the only curves in the plane with constant curvature are lines and circles. The case of lines occurs precisely when the curvature is zero:

Exercise 1. Show that the only curves with constant zero curvature in $\mathbf{R}^n$ are straight lines. (*Hint*: We may assume that our curve, $\alpha\colon I \to \mathbf{R}^n$ has unit speed. Then $\kappa = \|\alpha''\|$. So zero curvature implies that $\alpha'' = 0$. Integrating the last expression twice yields the desired result.)

So it remains to consider the case where we have a planar curve whose curvature is equal to some nonzero constant c. We claim that in this case the curve has to be a circle of radius $1/c$. To this end we introduce the following definition. If a curve $\alpha\colon I \to \mathbf{R}^n$ has nonzero curvature, the *principal normal* vector field of α is defined as

$$N(t) := \frac{T'(t)}{\|T'(t)\|},$$

where $T(t) := \alpha'(t)/\|\alpha'(t)\|$ is the tantrix of α as we had defined earlier. Thus the principal normal is the tantrix of the tantrix.

Exercise 2. Show that $T(t)$ and $N(t)$ are orthogonal. (*Hint*: Differentiate both sides of the expression $\langle T(t), T(t)\rangle = 1$).

So, if α is a planar curve, $\{T(t), N(t)\}$ form a *moving frame* for $\mathbf{R}^2$, i.e., any element of $\mathbf{R}^2$ may be written as a linear combination of $T(t)$ and $N(t)$ for any choice of t. In particular, we may express the derivatives of T and

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N in terms of this frame. The definition of N already yields that, when α is parametrized by arclength,

$$T'(t) = \kappa(t)N(t).$$

To get the corresponding formula for N', first observe that

$$N'(t) = aT(t) + bN(t).$$

for some a and b. To find a note that, since $\langle T, N\rangle = 0$, $\langle T', N\rangle = -\langle T, N'\rangle$. Thus

$$\alpha = \langle N'(t), T(t)\rangle = -\langle T'(t), N(t)\rangle = -\kappa(t).$$

Exercise 3. Show that $b = 0$. (*Hint* Differentiate $\langle N(t), N(t)\rangle = 1$).

So we conclude that

$$N'(t) = -\kappa(t)T(t),$$

where we still assume that t is the arclength parameter. The formulas for the derivative may be expressed in the matrix notation as

$$\begin{bmatrix} T(t) \\ N(t) \end{bmatrix}' = \begin{bmatrix} \kappa(t) & 0 \\ 0 & -\kappa(t) \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \end{bmatrix}.$$

Now recall that our main aim here is to classify curves of constant curvature in the plane. To this end define the *center of the osculating circle* of α as

$$p(t) := \alpha(t) + \frac{1}{\kappa(t)}N(t).$$

The circle which is centered at $p(t)$ and has radius of $1/\kappa(t)$ is called the *osculating* circle of α at time t. This is the circle which best approximates α up to the second order:

Exercise 4. Check that the osculating circle of α is tangent to α at $\alpha(t)$ and has the same curvature as α at time t.

Now note that if α is a circle, then it coincides with its own osculating circle. In particular $p(t)$ is a fixed point (the center of the circle) and $\|\alpha(t) - p(t)\|$ is constant (the radius of the circle). Conversely:

Exercise 5. Show that if α has constant curvature c, then (i) $p(t)$ is a fixed point, and (ii) $\|\alpha(t) - p(t)\| = 1/c$ (*Hint*: For part (i) differentiate $p(t)$; part (ii) follows immediately from the definition of $p(t)$).

So we conclude that a curve of constant curvature $c \neq 0$ lies on a circle of radius $1/c$.

1.5 Signed Curvature and Turning Angle

As we mentioned earlier the curvature of a curve is a measure of how fast it is turning. When the curve lies in a plane, we may assign a sign of plus or minus one to this measure depending on whether the curve is rotating clockwise or counterclockwise. Thus we arrive at a more descriptive notion of curvature for planar curves which we call *signed curvature* and denote by $\overline{\kappa}$. Then we may write

$$|\overline{\kappa}| = \kappa.$$

To obtain a formula for $\overline{\kappa}$, for any vector $v \in \mathbf{R}^2$, let iv be the clockwise rotation by 90 degrees. Then we may simply set

$$\overline{\kappa}(t) := \frac{\big\langle T'(t), iT(t)\big\rangle}{\|\alpha'(t)\|}.$$

Exercise 6. Show that if α is a unit speed curve then

$$\overline{\kappa}(t) = \kappa(t)\big\langle N(t), iT(t)\big\rangle.$$

In particular, $|\overline{\kappa}| = \kappa$.

Exercise 7. Compute the signed curvatures of the clockwise circle $\alpha(t) = (\cos t, \sin t)$, and the counterclockwise circle $\alpha(t) = (\cos(-t), \sin(-t))$.

Another simple and useful way to define the signed curvature (and the regular curvature) of a planar curve is in terms of the *turning angle* θ, which is defined as follows. We claim that for any planar curve $\alpha\colon I \to \mathbf{R}^2$ there exists a function $\theta\colon I \to \mathbf{R}^2$ such that

$$T(t) = \big(\cos\theta(t), \sin\theta(t)\big).$$

Then, assuming that t is the arclength parameter, we have

$$\overline{\kappa}(t) = \theta'(t).$$

Exercise 8. Check the above formula.

Now we check that θ indeed exists. To this end note that T may be thought of as a mapping from I to the unit circle $\mathbf{S}^1$. Thus it suffices to show that

Proposition 9. *Show that for any continuous function $T\colon I \to \mathbf{S}^1$, where $I = [a, b]$ is a compact interval, there exists a continuous function $\theta\colon I \to \mathbf{S}^1$ such that the above formula relating T and θ holds.*

Proof. Since T is continuous and I is compact, T is *uniformly* continuous, this means that for $\epsilon > 0$, we may find a $\delta > 0$ such that $\|T(t) - T(s)\| < \epsilon$, whenever $|t - s| < \delta$. In particular, we may set δ_0 to be equal to some constant less than one, and ϵ_0 to be the corresponding constant. Now choose the points

$$a =: x_0 \leq x_1 \leq \cdots \leq x_n := b$$

such that $|x_i - x_{i-1}| < \epsilon_0$, for $i = 1, \ldots, n$. Then T restricted to each subinterval $[x_i, x_{i-1}]$ is not unto. So we may define $\theta_i\colon [x_{i-1}, x_i] \to \mathbf{R}$ by setting $\theta_i(x)$ to be the angle in $[0, 2\pi)$, measured counterclockwise, between $T(x_{i-1})$ and $T(x)$. Finally, θ may be defined as

$$\theta(x) := \theta_0 + \sum_{i=1}^{k-1} \theta_i(x_i) + \theta_k(x) \quad \text{if} \quad x \in [x_{k-1}, x_k].$$

$\square$

1.6 Total Signed Curvature and Winding Number

The *total signed curvature* of $\alpha\colon I \to \mathbf{R}^n$ is defined as

$$\overline{\tau}[\alpha] := \int_I \overline{\kappa}(t)\, dt$$

where t is the arclength parameter. Note that since $\overline{\kappa} = \theta'$, the fundamental theorem of calculus yields that, if $I = [a, b]$, then

$$\overline{\tau}[\alpha] = \theta(a) - \theta(b).$$

We say that $\alpha\colon [a, b] \to \mathbf{R}^2$ is a closed curve provided that $\alpha(a) = \alpha(b)$ and $T(a) = T(b)$.

Exercise 10. Show that the total signed curvature of a closed curve is a multiple of 2π.

So, if α is a closed curve,

$$\operatorname{rot}[\alpha] := \frac{1}{2\pi}\int_I \overline{\kappa}(t)\,dt$$

is an integer which we call the Hopf *rotation index* or *winding number* of α. So we have

$$\tau[\alpha] = \operatorname{rot}[\alpha] 2\pi.$$

Exercise 11. (i) Compute the total curvature and rotation index of a circle which has been oriented clockwise, and a circle which is oriented counterclockwise. Sketch the *figure eight curve* $(\cos t, \sin 2t)$, $0 \leq t \leq 2\pi$, and compute its total signed curvature and rotation index.

We say that α is simple if it is one-to-one in the interior of I. The following result proved by H. Hopf is one of the fundamental theorems in theory of planar curves.

Theorem 12. *Any simple closed planar curve has rotation index* ± 1.

Hopf proved the above result using analytic methods including the Green's theorem. Here we outline a more elementary proof which will illustrate that the above theorem is simply a generalization of one of the most famous result in classical geometry: the sum of the angles in a triangle is π, which is equivalent to the sum of the exterior angles being 2π.

First we will give another definition for $\overline{\tau}$ which will establish the connection between the total signed curvature and the sum of the exterior angles in a *polygon*. By a polygon we mean an ordered set of points

$$P := \big(p_0, \dots, p_n\big)$$

in $\mathbf{R}^2$, where $p_n = p_0$, but $p_i \neq p_{i-1}$, for $i = 1, \dots, n$. Each p_i is called a vertex of P. At each vertex p_i, $i = 1 \dots n$, we define the *exterior angle* θ_i to be the angle in $[-\pi, \pi]$ determined by the vectors $p_i - p_{i-i}$, and $p_{i+1} - p_i$, and measured in the counterclockwise direction (we set $p_{n+1} := p_1$). The total curvature of P is defined as the sum of these angles:

$$\overline{\tau}[P] := \sum_{i=1}^{n} \theta_i.$$

Now let $\alpha\colon [a,b] \to \mathbf{R}^2$ be a closed planar curve. For $i = 0, \dots, n$, set

$$t_i := a + i\frac{b-a}{n},$$

and let

$$P_n[\alpha] := \big(\alpha(t_0), \dots, \alpha(t_n)\big)$$

be the n^{th} polygonal approximation of α. The following proposition shows that the total curvature of a closed curve is just the limit of the sum of the exterior angles of the polygonal approximations.

Proposition 13.

$$\overline{\tau}[\alpha] = \lim_{n\to\infty} \overline{\tau}\big[P_n[\alpha]\big].$$

Proof. Let θ be the rotation angle of α, and θ_i be the exterior angles of $P_n[\alpha]$. If we choose n large enough, then there exists, for $i = 0, \dots, n$, an element $\overline{t}_i \in [t_{i-1}, t_i]$ such that $T(t_i)$ is parallel to $\alpha(t_i) - \alpha(t_{i-1})$. Consequently

$$\theta_i = \theta(\overline{t}_i) - \theta(\overline{t}_{i-1}).$$

By the mean value theorem, there exists $t_i^* \in [\overline{t}_{i-1}, \overline{t}_i]$ such that

$$\theta(\overline{t}_i) - \theta(\overline{t}_{i-1}) = \theta'(t_i^*)(\overline{t}_i - \overline{t}_{i-1}) = \kappa(t_i^*)(\overline{t}_i - \overline{t}_{i-1}).$$

So

$$\lim_{n\to\infty} \overline{\tau}\big[P_n[\alpha]\big] = \lim_{n\to\infty} \sum_{i=1}^{n} \theta_i = \lim_{n\to\infty} \sum_{i=1}^{n} \theta'(t_i^*)(\overline{t}_i - \overline{t}_{i-1}) = \int_a^b \overline{\kappa}(t)\, dt = \overline{\tau}[\alpha].$$

□

Exercise* 14. Verify the second statement in the proof of the above theorem.

Now to complete the proof of Theorem 12 we need to verify:

Exercise* 15. Show that any simple polygon with more than three vertices has a vertex such that if we delete that vertex then the remaining polygon is still simple.

Exercise* 16. Show that the operation of deleting the vertex of a polygon described above does not change the sum of the exterior angles.

Since the sum of the exterior angles in a triangle is 2π, it would follow then that the sum of the exterior angles in any simple polygon is 2π. This in turn would imply Theorem 12 via Proposition 13.

1.7 The fundamental theorem for planar curves

If $\alpha\colon [0, L] \to \mathbf{R}^2$ is a planar curve parametrized by arclength, then its signed curvature yields a function $\overline{\kappa}\colon [0, L] \to \mathbf{R}$. Now suppose that we are given a continuous function $\overline{\kappa}\colon [0, L] \to \mathbf{R}$. Is it always possible to find a unit speed curve $\alpha\colon [0, L] \to \mathbf{R}^2$ whose signed curvature is $\overline{\kappa}$? If so, to what extent is such a curve unique? In this section we show that the signed curvature does indeed determine a planar curve, and such a curve is unique up to rigid motion.

By a rigid motion we mean a composition of a translation with a rotation. A translation is a mapping $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$T(p) := p + v$$

where v is a fixed vector. And a rotation $\rho\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a linear mapping given by

$$\rho\Big(\begin{bmatrix} x \\ y \end{bmatrix}\Big) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Exercise 17. Show that the signed curvature of a planar curve is invariant under rigid motions.

Exercise 18. Show that if the curvature of a planar curve $\alpha\colon I \to \mathbf{R}^2$ does not vanish at an interior point t_0 of I then there exists an open neighborhood U of t_0 in I such that $\alpha(U)$ lies on one side of the tangent line of α at t_0. (*Hint*: By the invariance of signed curvature under rigid motions, we may assume that $\alpha(t_0) = (0, 0)$ and $\alpha'(0) = (1, 0)$. Then we may reparametrize α as $(t, f(t))$ in a neighborhood of t_0. Recalling the formula for curvature for graphs, and applying the Taylor's theorem yields the desired result.)

Now suppose that we are given a function $\overline{\kappa}\colon [0, L] \to \mathbf{R}$. If there exist a curve $\alpha\colon [0, L] \to \mathbf{R}^2$ with signed curvature $\overline{\kappa}$, then

$$\theta' = \overline{\kappa}$$

where θ is the rotation angle of α. Integration yields

$$\theta(t) := \int_0^t \overline{\kappa}(s)ds + \theta(0).$$

By the definition of the turning angle

$$\alpha'(t) = \Big(\cos\theta(t), \sin\theta(t)\Big).$$

Consequently,

$$\alpha(t) = \Big(\int_0^t \cos\theta(s)ds, \int_0^t \sin\theta(s)ds \Big) + \alpha(0),$$

which gives an explicit formula for the desired curve.

Exercise 19. Let α, $\beta\colon [0, L] \to \mathbf{R}^2$ be unit speed planar curves with the same signed curvature function $\overline{\kappa}$. Show that there exists a rigid motion $m\colon \mathbf{R}^2 \to \mathbf{R}^2$ such that $\alpha(t) = m(\beta(t))$.

Exercise 20. Use the above formula to show that the only closed curves of constant curvature in the plane are circles.