Math 2411 May 5, 2006
Honors Calculus III
Spring 2006, Georgia Tech

Final Exam

Time: 180min

1. Show that the midpoints of any quadrilateral determine a parallelogram.

2. Find a formula for the length of a curve given by $r = f(\theta)$ in polar coordinates. (*Hint*: this curve may be parametrized as $(f(\theta)\cos(\theta), f(\theta)\sin(\theta))$.

3. Show that the orbit of a planet always lies in a plane which passes through the sun (*Hint*: Let $r(t)$ be the position vector of the planet, with respect to the sun and differentiate $r \times r'$).

4. Find the maximum and minimum values of $f(x, y) = x^2 + xy + y^2$ on the disk $x^2 + y^2 \leq 4$.

5. Show that for any n positive real numbers $x_1, \ldots x_n$,

$$\sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

(*Hint*: Maximize $x_1^2 \ldots x_n^2$ subject to the constraint $x_1^2 + x_2^2 + \cdots + x_n^2 = a^2$.)

6. Show that the area of a surface given by rotating the graph of a function $f(x)$, $a \leq x \leq b$, around the x-axis is given by $2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$. Use this result to show that the area of a sphere cut by a pair of parallel planes depends only on the distance between the two planes.

7. Find the average height (i.e., z-coordinate) of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 1$).

8. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using Green's theorem.

9. Show that if B is a solid in R^3, then

$$Volume(B) = \frac{1}{3}\int_{\partial B} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$$

10. Evaluate $\int_X \omega$ where X is the helicoid

$$X(s,t) = (s\cos t, s\sin t, t), \quad 0 \leq s \leq 1, \quad 0 \leq t \leq 4\pi$$

and $\omega = zdx \wedge dy + 3dz \wedge dx - xdy \wedge dz$.

11 (Extra Credit). Show that there is no force of gravity inside a hollow planet.

Each problem is worth 10pts.