Math 23B
Multivariable Calculus
Fall 1999, UCSC

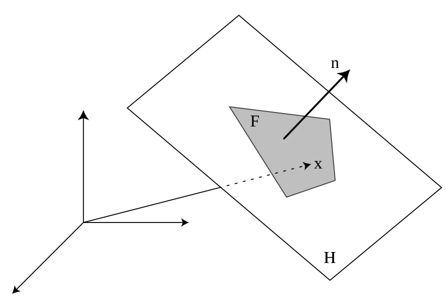
Nov 23, 1999

PRACTICE QUIZ 4

1. Use Gauss's theorem to find a relation between the volume of a polyhedron and the area of its faces.

By a *polyhedron* S, we mean a closed oriented surface (such as a cube or tetrahedron) which is made up of finitely many polygons glued together along their edges. Each polygon is called a *face* of S. Suppose that S has k faces which we denote by F_i. Let $\mathbf{n}_i$ be the outward unit normal to the face F_i.

Step (i) Show that for any point $\mathbf{x}$ in F_i, the quantity $d_i := \mathbf{x} \cdot \mathbf{n}_i$ is constant. What is the meaning of d_i? When is it positive? When is it negative? And when is it zero? (Note: If H_i denotes the plane of F_i, then d_i is called the signed distance of H_i from the origin.)



Step (ii) Let $\mathbf{r}(x, y, z) := (x, y, z)$. $\mathbf{r}$ is called the *position vector field.* Compute the flux of the position vector field across S; show that

$$\int\int_S \mathbf{r} \cdot \mathbf{n} \, dS = \sum_{i=1}^{k} Area(F_i) \, d_i.$$

Step (iii) Compute the divergence of $\mathbf{r}$. Show that

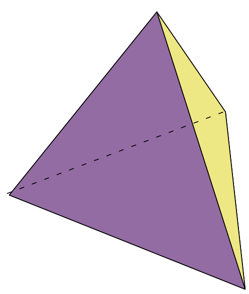
$$\int\int\int_B \nabla \cdot \mathbf{r}\, dV = 3\, Volume(B),$$

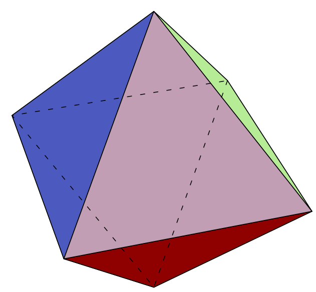
where B denotes the region bounded by S.

Step(iv) Use Gauss's theorem (Total Divergence = Flux) to conclude that

$$Volume(B) = \frac{1}{3}\sum_{i=1}^{k} Area(F_i)\, d_i.$$

2. (a) Use the result of the previous problem to compute the volume of a cube. (b) Find the volumes of a regular tetrahedron, and a regular octahedron.





3 (Extra Credit). Find the volumes of the regular icosahedron and dodecahedron.