Math 142 Oct 19, 2001
Calculus II
Fall 2001, USC

Solutions to Midterm 2

1a. To differentiate $y = \cos x^{\sin x}$, first take natural log of both sides:

$$\ln y = \sin x \ln(\cos x).$$

Then differentiate both sides:

$$\begin{aligned} \frac{y'}{y} &= \sin' x \ln(\cos x) + \sin x(\ln(\cos x))' \\ &= \cos x \ln(\cos x) + \sin x \left(\frac{1}{\cos x}(-\sin x) \right). \end{aligned}$$

Finally, multiply both sides by y:

$$y' = \cos x^{\sin x}(\cos x \ln(\cos x) - \sin x \tan x).$$

1b. See the solution set to Midterm 1.

2a. To find $\int \ln x \, dx$, let

$$u = \ln x \qquad \text{and} \qquad dv = dx.$$

Then

$$du = \frac{1}{x} dx \qquad \text{and} \qquad v = x.$$

So integration by parts yields:

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int \frac{x}{x} dx \\ &= x \ln x - x + C \end{aligned}$$

2b. To find $\int \sin^n x \, dx$, where n is an odd integer, write the integrand as $\sin^{n-1} x \sin x$, and use the formula $\sin^2 x + \cos^2 x = 1$:

$$\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx$$

Let $u = \cos x$, then $du = -\sin x\,dx$. So the above integral becomes:

$$\int (1-u^2)(-du) = -u + \frac{1}{3}u^3 + C = -\cos x + \frac{1}{3}\cos^3 x + C.$$

2d.

$$\frac{x-7}{x^2-x-12} = \frac{x-7}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3} = \frac{A(x+3)+B(x-4)}{(x-4)(x+3)}$$

So it follows that

$$x - 7 = A(x+3) + B(x-4).$$

Setting $x = 4$ on both sides of the above equation, we get

$$4 - 7 = A(4+3) + B(4-4).$$

So $-3 = 7A$, which yields $A = -3/7$. Similarly, setting $x = -3$, we get

$$-3 - 7 = A(-3+3) + B(-3-4),$$

which yields that $-10 = -7B$, or $B = 10/7$. So

$$\begin{aligned}\int \frac{x-7}{x^2-x-12}\,dx &= \int \frac{-3/7}{x-4}\,dx + \int \frac{10/7}{x+3}\,dx \\ &= \frac{-3}{7}\ln(x-4) + \frac{10}{7}\ln(x+3) + C.\end{aligned}$$

2c. Let $u = 1 - x$, then $du = -dx$. So

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x}} &= \int_{1-0}^{1-1} u^{-1/2}(-du) \\ &= -\int_1^0 u^{-1/2}du \\ &= \frac{-1}{-1/2+1}u^{-1/2+1}\Big|_1^0 = 2.\end{aligned}$$

3a. $\lim_{x\to 0}(\cos x)^{\frac{1}{x}}$ is indeterminate of the form 1^∞. So proceed as follows:

$$\begin{aligned} y &= \cos x^{\frac{1}{x}} \\ \ln y &= \frac{1}{x}\ln\cos x \\ \lim_{x\to 0}\ln y &= \lim_{x\to 0}\frac{1}{x}\ln\cos x. \end{aligned}$$

Since the last limit above is indeterminate of the form $0/0$, we may apply the L'Hopital's rule:

$$\lim_{x\to 0}\frac{\ln\cos x}{x}\ln\cos x = \lim_{x\to 0}\frac{-\sin x/\cos x}{1} = \frac{0/1}{1} = 0.$$

So $\lim_{x\to 0}\ln y = 0$. But, since ln is continuous, $\lim_{x\to 0}\ln y = \ln(\lim_{x\to 0} y)$. So $\ln(\lim_{x\to 0} y) = 0$, which yields that

$$\lim_{x\to 0} y = e^0 = 1.$$

3b. $\lim_{x\to 0}(x\ln x^2)$ is indeterminate of the form $0\cdot\infty$, so we proceed as follows:

$$\lim_{x\to 0}(x\ln x^2) = \lim_{x\to 0}\frac{\ln x^2}{1/x}.$$

Now the limit on the right is of the form ∞/∞, so we may apply the L'Hopital's rule:

$$\lim_{x\to 0}\frac{\ln x^2}{1/x} = \lim_{x\to 0}\frac{2x/x^2}{-1/x^2} = \lim_{x\to 0} -2x = 0.$$

So we conclude that a_n converges.L

4a. The general term for the series $\frac{-1}{4}$, $\frac{2}{8}$, $\frac{-3}{16}$, $\frac{4}{32}$, $\frac{-5}{64}, \ldots$ is given by

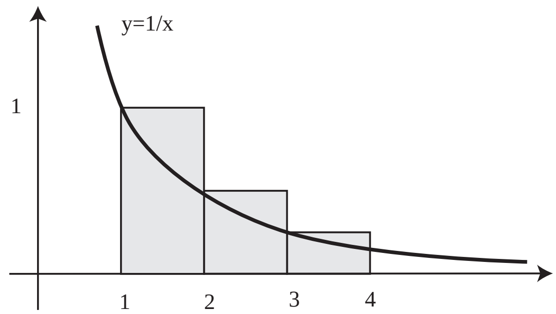
$$a_n = (-1)^n\frac{n}{2^{n+1}}.$$

Since $-|a_n| \le a_n \le |a_n|$, then a_n converges if and only if $|a_n|$ converges. The latter limit may be computed using the L'Hopital's rule:

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{n}{2^{n+1}} = \lim_{n\to\infty}\frac{1}{2^{n+1}\ln 2} = \frac{1}{\infty} = 0.$$

So we conclude that a_n converges.

4b. Consider the following picture: Since the area under the graph of



$y = 1/x$, from 1 to n, is less than the sum of the areas of the first $n-1$ rectangles, we have

$$\ln n = \int_1^n \frac{1}{x}\,dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}.$$

Since $\lim_{n\to\infty} \ln n = \infty$, it follows that the series on the right hand side of the above inequality diverges.