Math 121B
Differential Geometry and Topology
Winter 2000, UCSC

Feb 14, 2000[1]

Lecture Notes 4

3.2 Ratio of areas

In the previous subsection we gave a geometric interpretation for the sign of Gaussian curvature. Here we describe the geometric significance of the magnitude of K.

If V is a sufficiently small neighborhood of p in M (where M, as always, denotes a regular embedded surface in $\mathbf{R}^3$), then it is easy to show that there exist a patch (U, X) centered at p such that $X(U) = V$. Area of V is then defined as follows:

$$\text{Area}(V) := \int\int_U \|D_1X \times D_2X\|\, du^1du^2.$$

Using the chain rule, one can show that the above definition is independent of the the patch.

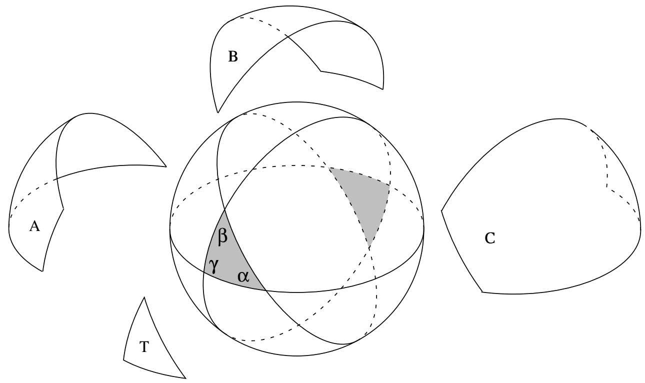
Exercise 3.2.1. Let $V \subset \mathbf{S}^2$ be a region bounded in between a pair of great circles meeting each other at an angle of α. Show that $\text{Area}(V) = 2\alpha$(*Hints*: Let $U := [0, \alpha] \times [0, \pi]$ and $X(\theta, \phi) := (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. Show that $\|D_1X \times D_2X\| = |\sin\phi|$. Further, note that, after a rotation we may assume that $X(U) = V$. Then an integration will yield the desired result).

Exercise 3.2.2. Use the previous exercise to show that the area of a geodesic triangle $T \subset \mathbf{S}^2$ (a region bounded by three great circles) is equal to sum of its angles minus π (*Hints:* Use the picture below: $A + B + C + T = 2\pi$, and $A = 2\alpha - T$, $B = 2\beta - T$, and $C = 2\gamma - T$).

Let $V_r := B_r(p) \cap M$. Then, if r is sufficiently small, $V(r) \subset X(U)$, and, consequently, $U_r := X^{-1}(V_r)$ is well defined. In particular, we may compute the area of V_r using the patch (U_r, X). In this section we show that

$$|K(p)| = \lim_{r\to 0} \frac{\text{Area}(n(V_r))}{\text{Area}(V_r)}.$$

[1]Last revised: February 17, 2000



Exercise 3.2.3. Recall that the mean value theorem states that $\int\int_U f du^1 du^2 = f(\bar{u}^1, \bar{u}^2)\,\text{Area}(U)$, for some $(\bar{u}^1, \bar{u}^2) \in U$. Use this theorem to show that

$$\lim_{r\to 0} \frac{\text{Area}(n(V_r))}{\text{Area}(V_r)} = \frac{\|D_1 N(0,0) \times D_2 N(0,0)\|}{\|D_1 X(0,0) \times D_2 X(0,0)\|}$$

(Recall that $N := n \circ X$.)

Exercise 3.2.4. Prove Lagrange's identity: for every pair of vectors $v, w \in \mathbf{R}^3$,

$$\|v \times w\|^2 = \det \begin{vmatrix} \langle v, v\rangle & \langle v, w\rangle \\ \langle w, v\rangle & \langle w, w\rangle \end{vmatrix}.$$

Now set $g(u^1, u^2) := \det[g_{ij}(u^1, u^2)]$. Then, by the previous exercise it follows that $\|D_1 X(0,0) \times D_2 X(0,0)\| = \sqrt{g(0,0)}$. Hence, to complete the proof of the main result of this section it remains to show that

$$\|D_1 N(0,0) \times D_2 N(0,0)\| = K(p)\sqrt{g(0,0)}.$$

We prove the above formula using two different methods:
METHOD 1. Recall that $K(p) := \det(S_p)$, where $S_p := -dn_p \colon T_pM \to T_pM$ is the shape operator of M at p. Also recall that $D_iX(0,0)$, $i = 1, 2$, form a basis for T_pM. Let S_{ij} be the coefficients of the matrix representation of S_p with respect to this basis, then

$$S_p(D_iX) = \sum_{j=1}^{2} S_{ij}\, D_jX.$$

Further, recall that $N := n \circ X$. Thus the chain rule yields:

$$S_p(D_iX) = -dn(D_iX) = -D_i(n \circ X) = -D_iN.$$

Exercise 3.2.5. Verify the middle step in the above formula, i.e., show that $dn(D_iX) = D_i(n \circ X)$.

From the previous two lines of formulas, it now follows that

$$-D_iN = \sum_{j=1}^{2} S_{ij}\, D_jX.$$

Taking the inner product of both sides with D_kN, $k = 1$, 2, we get

$$\langle -D_iN, D_kN\rangle = \sum_{j=1}^{2} S_{ij}\langle D_jX, D_kN\rangle.$$

Exercise 3.2.6. Let F, $G\colon U \subset \mathbf{R}^2 \to \mathbf{R}^3$ be a pair of mappings such that $\langle F, G\rangle = 0$. Prove that $\langle D_iF, G\rangle = -\langle F, D_iG\rangle$.

Now recall that $\langle D_jX, N\rangle = 0$. Hence the previous exercise yields:

$$\langle D_jX, D_kN\rangle = -\langle D_{kj}X, N\rangle = -l_{ij}.$$

Combining the previous two lines of formulas, we get: $\langle D_iN, D_kN\rangle = \sum_{k=1}^{2} S_{ij}l_{jk}$; which in matrix notation is equivalent to

$$[\langle D_iN, D_jN\rangle] = [S_{ij}][l_{ij}].$$

Finally, recall that $\det[\langle D_iN, D_kN\rangle] = \|D_1N \times D_2N\|^2$, $\det[S_{ij}] = K$, and $\det[l_{ij}] = Kg$. Hence taking the determinant of both sides in the above equation, and then taking the square root yields the desired result.

Next, we discuss the second method for proving that $\|D_1N \times D_2N\| = K\sqrt{g}$.

METHOD 2. Here we work with a special patch which makes the computations easier:

Exercise 3.2.7. Show that there exist a patch (U, X) centered at p such that $[g_{ij}(0,0)]$ is the identity matrix. (*Hint:* Start with a Monge patch with respect to T_pM)

Thus, if we are working with the coordinate patch referred to in the above exercise, $g(0,0) = 1$, and, consequently, all we need is to prove that $\|D_1N(0,0) \times D_2N(0,0)\| = K(p)$.

Exercise 3.2.8. Let $f\colon U \subset \mathbf{R}^2 \to \mathbf{S}^2$ be a differentiable mapping. Show that $\langle D_i f(u^1, u^2), f(u^1, u^2)\rangle = 0$ (*Hints:* note that $\langle f, f\rangle = 1$ and differentiate).

It follows from the previous exercise that $\langle D_i N, N\rangle = 0$. Now recall that $N(0,0) = n \circ X(0,0) = n(p)$. Hence, we may conclude that $N(0,0) \in T_pM$. Further recall that $\{D_1X(0,0), D_2X(0,0)\}$ is now an orthonormal basis for T_pM (because we have chosen (U,X) so that $[g_{ij}(0,0)]$ is the identity matrix). Consequently,

$$D_iN = \sum_{k=1}^{2} \langle D_iN, D_kX\rangle D_kX,$$

where we have omitted the explicit reference to the point $(0,0)$ in the above formula in order to make the notation less cumbersome (it is important to keep in mind, however, that the above is valid only at $(0,0)$). Taking the inner product of both sides of this equation with $D_jN(0,0)$ yields:

$$\langle D_iN, D_jN\rangle = \sum_{k=1}^{2} \langle D_iN, D_kX\rangle\langle D_kX, D_jN\rangle.$$

Now recall that $\langle D_iN, D_kX\rangle = -\langle N, D_{ij}X\rangle = -l_{ij}$. Similarly, $\langle D_kX, D_jN\rangle = -l_{kj}$. Thus, in matrix notation, the above formula is equivalent to the following:

$$[\langle D_iN, D_jN\rangle] = [l_{ij}]^2$$

Finally, recall that $K(p) = \det[l_{ij}(0,0)]/\det[g_{ij}(0,0)] = \det[l_{ij}(0,0)]$. Hence, taking the determinant of both sides of the above equation yields the desired result.

3.3 Product of principal curvatures

For every $v \in T_pM$ with $\|v\| = 1$ we define the *normal curvature* of M at p in the direction of v by

$$k_v(p) := \langle \gamma''(0), n(p)\rangle,$$

where $\gamma\colon (-\epsilon, \epsilon) \to M$ is a curve with $\gamma(0) = p$ and $\gamma'(0) = v$.

Exercise 3.3.1. Show that $k_v(p)$ does not depend on γ.

In particular, by the above exercise, we may take γ to be a curve which lies in the intersection of M with a plane which passes through p and is normal to $n(p) \times v$. So, intuitively, $k_v(p)$ is a measure of the curvature of an orthogonal cross section of M at p.

Let $UT_pM := \{v \in T_pM \mid \|v\| = 1\}$ denote the *unit tangent space* of M at p. The *principal curvatures* of M at p are defined as

$$k_1(p) := \min_v k_v(p), \quad \text{and} \quad k_2(p) := \max_v k_v(p),$$

where v ranges over UT_pM. Our main aim in this subsection is to show that

$$K(p) = k_1(p)k_2(p).$$

Since $K(p)$ is the determinant of the shape operator S_p, to prove the above it suffices to show that $k_1(p)$ and $k_2(p)$ are the eigenvalues of S_p.

First, we need to define the *second fundamental form* of M at p. This is a bilinear map $\mathrm{II}_p \colon T_pM \times T_pM \to \mathbf{R}$ defined by

$$\mathrm{II}_p(v, w) := \langle S_p(v), w \rangle.$$

We claim that, for all $v \in UT_pM$,

$$k_v(p) = \mathrm{II}_p(v, v).$$

The above follows from the following computation

$$\begin{aligned} \langle S_p(v), v \rangle &= -\langle dn_p(v), v \rangle \\ &= -\langle (n \circ \gamma)'(0), \gamma'(0) \rangle \\ &= \langle (n \circ \gamma)(0), \gamma''(0) \rangle \\ &= \langle n(p), \gamma''(0) \rangle \end{aligned}$$

Exercise 3.3.2. Verify the passage from the second to the third line in the above computation, i.e., show that $-\langle (n \circ \gamma)'(0), \gamma'(0) \rangle = \langle (n \circ \gamma)(0), \gamma''(0) \rangle$ (*Hint:* Set $f(t) := \langle n(\gamma(t)), \gamma'(t) \rangle$, note that $f(t) = 0$, and differentiate.)

So we conclude that $k_i(p)$ are the minimum and maximum of $\mathrm{II}_p(v)$ over UT_pM. Hence, all we need is to show that the extrema of II_p over UT_pM coincide with the eigenvalues of S_p.

Exercise 3.3.3. Show that II_p is symmetric, i.e., $\mathrm{II}_p(v, w) = \mathrm{II}_p(w, v)$ for all v, $w \in T_pM$.

By the above exercise, S_p is a self-adjoint operator, i.e, $\langle S_p(v), w\rangle = \langle v, S_p(w)\rangle$. Hence S_p is orthogonally diagonalizable, i.e., there exist orthonormal vectors $e_i \in T_pM$, $i = 1, 2$, such that

$$S_p(e_i) = \lambda_i e_i.$$

By convention, we suppose that $\lambda_1 \leq \lambda_2$. Now note that each $v \in UT_pM$ may be represented uniquely as $v = v^1e_1 + v^2e_2$ where $(v^1)^2 + (v^2)^2 = 1$. So for each $v \in UT_pM$ there exists a unique angle $\theta \in [0, 2\pi)$ such that

$$v(\theta) := \cos\theta e_1 + \sin\theta e_2;$$

Consequently, bilinearity of II_p yields

$$\mathrm{II}_p(v(\theta), v(\theta)) = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta.$$

Exercise 3.3.4. Verify the above claim, and show that minimum and maximum values of II_p are λ_1 and λ_2 respectively. Thus $k_1(p) = \lambda_1$, and $k_2(p) = \lambda_2$.

The previous exercise completes the proof that $K(p) = k_1(p)k_2(p)$, and also yields the following formula which was discovered by Euler:

$$k_v(p) = k_1(p)\cos^2\theta + k_2(p)\sin^2\theta.$$

In particular, note that by the above formula there exists always a pair of *orthogonal* directions where $k_v(p)$ achieves its maximum and minimum values. These are known as the *principal directions* of M at p.