Math 121B
Differential Geometry and Topology
Winter 2000, UCSC

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Lecture Notes 3

3 Meaning of Gaussian Curvature

In the previous lecture we gave a formal definition for Gaussian curvature K in terms of the differential of the gauss map, and also derived explicit formulas for K in local coordinates. In this lecture we explore the geometric meaning of K.

3.1 A measure for local convexity

Let $M \subset \mathbf{R}^3$ be a regular embedded surface, $p \in M$, and H_p be hyperplane passing through p which is parallel to T_pM. We say that M is *locally convex* at p if there exists an open neighborhood V of p in M such that V lies on one side of H_p. In this section we prove

Theorem 3.1.1. *If $K(p) > 0$ then M is locally convex at p, and if $K(p) < 0$ then M is not locally convex at p.*

In the case where $K(p) = 0$, we cannot in general draw any conclusion with regard to the local convexity of M at p as the following two exercises demonstrate:

Exercise 3.1.2. Show that there exists a surface M and a point $p \in M$ such that M is strictly locally convex at p; however, $K(p) = 0$ (*Hint:* Let M be the graph of the equation $z = (x^2 + y^2)^2$. Then M may be covered by the Monge patch $X(u^1, u^2) := (u^1, u^2, ((u^1)^2 + (u^2)^2)^2)$. Use the Monge Ampere equation derived in the previous lecture to compute the curvature at $X(0, 0)$.).

Exercise 3.1.3. Let M be the *monkey saddle*, i.e., the graph of the equation $z = y^3 - 3yx^2$, and $p := (0, 0, 0)$. Show that $K(p) = 0$, but M is not locally convex at p.

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After a rigid motion, we may assume that $p = (0,0,0)$ and T_pM is the xy-plane. Then, using the inverse function theorem, it is easy to show that there exist a monge patch (U, X) centered at p, as the following exercise demonstrates:

Exercise 3.1.4. Define $\pi\colon M \to \mathbf{R}^2$ by $\pi(q) := (q^1, q^2, 0)$. Show that π_{*_p} is locally one-to-one. Then, by the inverse function theorem, it follows that π is a local diffeomorphism. So there exist a neighborhood U of $(0,0)$ such that $\pi^{-1}\colon U \to M$ is one-to-one and smooth. Let $f(u^1, u^2)$ denote the z-coordinate of $\pi^{-1}(u^1, u^2)$, and set $X(u^1, u^2) := (u^1, u^2, f(u^1, u^2))$. Show that (U, X) is a proper regular patch.

The previous exercise shows that local convexity of M at p depends on whether or not f changes sign in a neighborhood of the origin. To examine this we need to recall the Taylor's formula for function of two variables:

$$f(u^1, u^2) = f(0,0) + \sum_{i=1}^{2} D_i f(0,0) u^i + \frac{1}{2} \sum_{i,j=1}^{2} D_{ij} f(\xi^1, \xi^2) u^i u^j,$$

where (ξ^1, ξ^2) is a point on the line segment between (u^1, u^2) and $(0,0)$.

Exercise 3.1.5. Prove the Taylor's formula given above (*Hints:* First recall Taylor's formula for functions of one variable: $g(t) = g(0) + g'(0)t + (1/2)g''(s)t^2$, where $s \in [0,t]$. Then define $\gamma(t) := (tu^1, tu^2)$, set $g(t) := f(\gamma(t))$, and apply Taylor's formula to g. The chain rule will yield the desired result.)

Next note that, by construction, $f(0,0) = 0$. Further $D_1 f(0,0) = 0 = D_2 f(0,0)$ as well. Thus it follows that

$$f(u^1, u^2) = \frac{1}{2} \sum_{i,j=1}^{2} D_{ij} f(\xi^1, \xi^2) u^i u^j.$$

Hence, to complete the proof of Theorem 3.1.1, it remains to show how the quantity on the right hand side of the above equation is influenced by $K(p)$. To this end, recall the Monge-Ampere equation for curvature:

$$\det(\operatorname{Hess} f(\xi^1, \xi^2)) = K(f(\xi^1, \xi^2))(\sqrt{1 + \|\operatorname{grad} f(\xi^1, \xi^2)\|^2})^2.$$

Now note that $K(f(0,0)) = K(p)$. Thus, by continuity, if U is a sufficiently small neighborhood of $(0,0)$, the sign of $\det(\text{Hess}\, f)$ agrees with the sign of $K(p)$ throughout U.

Finally, we need some basic facts about quadratic forms. A quadratic form is a function of two variables $Q\colon \mathbf{R}^2 \to \mathbf{R}$ given by

$$Q(x, y) := ax^2 + 2bxy + cy^2,$$

where a, b and c are constants. Q is said to be *definite* if $Q(x, x) \neq 0$ whenever $x \neq 0$.

Exercise 3.1.6. Show that if $ac - b^2 > 0$, then Q is definite, and if $ac - b^2 < 0$, then Q is not definite (*Hints:* For the first part, suppose that $x \neq 0$, but $Q(x, y) = 0$. Then $ax^2 + 2bxy + cy^2 = 0$, which yields $a + 2b(x/y) + c(x/y)^2 = 0$. Thus the discriminant of this equation must be positive, which will yield a contradiction. The proof of the second part is similar).

Theorem 3.1.1 follows from the above exercise.