Math 121B
Differential Geometry and Topology
Winter 2000, UCSC

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Lecture Notes 2

2 Gaussian Curvature

The principal geometric quantity associated to surfaces in $\mathbf{R}^3$ is that of their Gaussian curvature which we define in this lecture.

2.1 The tangent space

Let $M \subset \mathbf{R}^3$ be a regular embedded surface, as we defined in the previous lecture, and let $p \in M$. By the tangent space of M at p, denoted by T_pM, we mean the set of all vectors v in $\mathbf{R}^3$ such that for each vector v there exists a smooth curve $\gamma\colon (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$.

Exercise 2.1.1. Let $H \subset \mathbf{R}^3$ be a plane. Show that, for all $p \in H$, T_pH is the plane parallel to H which passes through the origin.

Exercise 2.1.2. Prove that, for all $p \in M$, T_pM is a 2-dimensional vector subspace of $\mathbf{R}^3$ (*Hint:* Let (U, X) be a proper regular patch centered at p, i.e., $X(0,0) = p$. Recall that $dX_{(0,0)}$ is a linear map and has rank 2. Thus it suffices to show that $T_pM = dX_{(0,0)}(\mathbf{R}^2)$).

Exercise 2.1.3. Prove that $D_1X(0,0)$ and $D_2X(0,0)$ form a basis for T_pM (*Hint:* Show that $D_1X(0,0) = dX_{(0,0)}(1,0)$ and $D_2X(0,0) = dX_{(0,0)}(0,1)$).

2.2 The local gauss map

By a local gauss map of M centered at p we mean a pair (V, n) where V is an open neighborhood of p in M and $n\colon V \to \mathbf{S}^2$ is a continuous mapping such that $n(p)$ is orthogonal to T_pM for all $p \in M$.

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Exercise 2.2.1. let (V, n) be a local gauss map of M centered at p. Show that $(V, -n)$ is also a local gauss map at p.

The above exercise shows that in general gauss map is not unique; however, given a local parameterization of the surface, we may define the local gauss map in a canonical way as described in the following exercise:

Exercise 2.2.2. Show that every $p \in M$ has an open neighborhood where the gauss map is well defined (*Hint:* Let (U, X) be a proper regular patch centered at p. Define $N\colon U \to \mathbf{S}^2$ by

$$N(u_1, u_2) := \frac{D_1X(u_1, u_2) \times D_2X(u_1, u_2)}{\|D_1X(u_1, u_2) \times D_2X(u_1, u_2)\|}.$$

Set $V := X(U)$, and recall that, since (U, X) is proper, V is open in M. Now define $n\colon V \to \mathbf{S}^2$ by

$$n(p) := N \circ X^{-1}(p).$$

Check that n is well-defined and is indeed the gauss map.

Exercise 2.2.3. Show that, for all $p \in \mathbf{S}^2$, $n(p) = p$ (*Hint:* Define $f\colon \mathbf{R}^3 \to \mathbf{R}$ by $f(p) := \|p\|^2$ and compute its gradient. Note that $\mathbf{S}^2$ is a level set of f. Thus the gradient of f at p must be orthogonal to $\mathbf{S}^2$).

2.3 Differential of a map between surfaces

Let M and N be regular embedded surfaces in $\mathbf{R}^3$ and $f\colon M \to N$ be a smooth map (recall from the first lecture that this means that f may be extended a smooth map in an open neighborhood of M). Then for every $p \in M$, we define a mapping $df_p\colon T_pM \to T_{f(p)}N$, known as the differential of M at p as follows. Let $v \in T_pM$ and let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a curve such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then we set

$$df_p(v) := (f \circ \gamma)'(0).$$

Exercise 2.3.1. Prove that df_p is well defined and linear (*Hint:* Let $\tilde{f}$ be a smooth extension of f to an open neighborhood of M. Then $d\tilde{f}_p$ is well defined. Show that for all $v \in T_pM$, $df_p(v) = d\tilde{f}_p(v)$.

2.4 The shape operator

let (V, n) be a local gauss map centered at $p \in M$. Then the shape operator of M at p with respect of n is defined as

$$S_p := -dn_p.$$

Note that the shape operator is determined up to two choices depending on the local gauss map.

Exercise 2.4.1. Show that S_p may be viewed as a linear operator on T_pM (*Hint:* By definition, S_p is a linear map from T_pM to $T_{n(p)}\mathbf{S}^2$. Thus it suffices to show that T_pM and $T_{f(p)}S^2$ are parallel).

Exercise 2.4.2. A subset V of M is said to be connected if any pairs of points p and q in V may be joined by a curve in V. Suppose that V is a connected open subset of M, and, furthermore, suppose that the shape operator vanishes throughout V, i.e., for every $p \in M$ and $v \in T_pM$, $S_p(v) = 0$. Show then that V must be flat, i.e., it is a part of a plane (*Hint:* It is enough to show that the gauss map is constant on V; or, equivalently, $n(p) = n(q)$ for all pairs of points p and q in V. Since V is connected, there exists a curve $\gamma\colon [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Furthermore, since V is open, we may choose γ to be smooth as well. Define $f\colon [0,1] \to \mathbf{R}$ by $f(t) := n \circ \gamma(t)$, and differentiate. Then $f'(t) = dn_{\gamma(t)}(\gamma'(t)) = 0$. Justify the last step and conclude that $n(p) = n(q)$.

Exercise 2.4.3. Compute the shape operator of a sphere of radius r (*Hint:* Define $\pi\colon \mathbf{R}^3 - \{0\} \to \mathbf{S}^2$ by $\pi(x) := x/\|x\|$. Note that π is a smooth mapping and $\pi = n$ on S^2 . Thus, for any $v \in T_p\mathbf{S}^2$, $d\pi_p(v) = dn_p(v)$).

2.5 Gaussian curvature

The Gaussian curvature of M at p is defined simply as the determinant of the shape operator:

$$K(p) := \det(S_p).$$

Exercise 2.5.1. Show that $K(p)$ does not depend on the choice of the local gauss map, i.e, replacing n by $-n$ does not effect the value of $K(p)$.

Exercise 2.5.2. Compute the curvature of a sphere of radius r (*Hint:* Use exercise 2.4.3).

2.6 An explicit formula in terms of local coordinates

Here we derive an explicit formula for $K(p)$ in terms of local coordinates. Let (U, X) be a proper regular patch centered at p. For $1 \leqslant i, j \leqslant 2$, define the functions $g_{ij} \colon U \to \mathbf{R}$ by

$$g_{ij}(u_1, u_2) := D_i X(u_1, u_2) \times D_j X(u_1, u_2).$$

Note that $g_{12} = g_{21}$. Thus the above defines three functions which are called the coefficients of the second fundamental form (a.k.a. the metric tensor) with respect to the given patch (U, X). In the classical notation, these functions are denoted by E, F, and G ($E := g_{11}$, $F := g_{12}$, and $G := g_{22}$). Next, define $l_{ij} \colon U \to \mathbf{R}$ by

$$l_{ij}(u_1, u_2) := \langle D_{ij} X(u_1, u_2), N(u_1, u_2) \rangle.$$

Thus l_{ij} is a measure of the second derivatives of X in a normal direction. l_{ij} are known as the coefficients of the second fundamental form of M with respect to the local patch (U, X) (the classical notation for these functions are $L := l_{11}$, $M := l_{12}$, and $N := l_{22}$). We claim that

$$K(p) = \frac{\det(l_{ij}(0,0))}{\det(g_{ij}(0,0))}.$$

To see the above, recall that $e_i(p) := D_i X(X^{-1}(p))$ form a basis for T_pM. Thus, since S_p is linear, $S_p(e_i) = \sum_{j=1}^{2} S_{ij} e_j$. This yields that $\langle S_p(e_i), e_k \rangle = \sum_{j=1}^{2} S_{ij} g_{jk}$. Suppose that

$$\langle S_p(e_i), e_k \rangle = l_{ik},$$

see the exercise below. Then we have $[l_{ij}] = [S_{ij}][g_{ij}]$, where the symbol $[\cdot]$ denotes the matrix with the given coefficients. Thus we can write $[S_{ij}] = [g_{ij}]^{-1}[l_{ij}]$ which yields the desired result.

Exercise 2.6.1. Show that $\langle S_p(e_i(p)), e_j(p) \rangle = l_{ij}(0,0)$ (*Hints:* note that $\langle n(p), e_j(p) \rangle = 0$ for all $p \in V$. Let $\gamma \colon (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = e_i(p)$. Define $f \colon (-\epsilon, \epsilon) \to M$ by $f(t) := \langle n(\gamma(t)), e_j(\gamma(t)) \rangle$, and compute $f'(0)$.)

Exercise 2.6.2. Let (U, X) be a Monge patch, i.e, $X(u_1, u_2) := (u_1, u_2, f(u_1, u_2))$, centered at $p \in M$. Show that

$$K(p) := \frac{\det(\operatorname{Hess} f(0,0))}{(1 + \| \operatorname{grad} f(0,0)\|^2)^2},$$

where $\operatorname{Hess} f := [D_{ij} f]$ is the Hessian matrix of f and $\operatorname{grad} f$ is its gradient.