Math 121B Jan 14, 2000[1]
Differential Geometry and Topology
Winter 2000, UCSC

Lecture Notes 1

1 Definition of a regular embedded surface

The main objects of study in this class are regular surfaces in 3-space, and Our main aim in this section is to give a precise and self-contained definition for a regular embedded surface. The only background we assume is some familiarity with elementary Calculus and Linear Algebra.

1.1 The Euclidean space

By $\mathbf{R}$ we shall always mean the set of real numbers. The set of all n-tuples of real numbers $\mathbf{R}^n := \{(p^1, \dots, p^n) \mid p^i \in \mathbf{R}\}$ is called the Euclidean n-space. So we have

$$p \in \mathbf{R}^n \iff p = (p^1, \dots, p^n), \quad p^i \in \mathbf{R}.$$

Let p and q be a pair of points (or vectors) in $\mathbf{R}^n$. We define $p + q := (p^1 + q^1, \dots, p^n + q^n)$. Further, for any scalar $r \in \mathbf{R}$, we define $rp := (rp^1, \dots, rp^n)$. It is easy to show that the operations of addition and scalar multiplication that we have defined turn $\mathbf{R}^n$ into a vector space over the field of real numbers. Next we define the standard inner product on $\mathbf{R}^n$ by

$$\langle p, q \rangle = p^1 q^1 + \dots + p^n q^n.$$

Note that the mapping $\langle \cdot, \cdot \rangle \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is linear in each variable and is symmetric. The standard inner product induces a norm on $\mathbf{R}^n$ defined by

$$\|p\| := \langle p, p \rangle^{1/2}.$$

If $p \in \mathbf{R}$, we usually write $|p|$ instead of $\|p\|$.

[1] Last revised: February 17, 2000

Exercise 1.1.1. (The Cauchy-Schwartz inequality) Prove that $|\langle p,q\rangle| \leqslant \|p\|\,\|q\|$, for all p and q in $\mathbf{R}^n$ (*Hints:* If p and q are linearly dependent the solution is clear. Otherwise, let $f(\lambda) := \langle p-\lambda q, p-\lambda q\rangle$. Then $f(\lambda) > 0$. Further, note that $f(\lambda)$ may be written as a quadratic equation in λ. Hence its discriminant must be negative).

The standard Euclidean distance in $\mathbf{R}^n$ is given by

$$\mathrm{dist}(p,q) := \|p-q\|.$$

Note that $(\mathbf{R}^n, \mathrm{dist})$ is a metric space. This means that (i) $\mathrm{dist}(p,q) \geqslant 0$, with equality if and only if $p = q$, (ii) $\mathrm{dist}(p,q) = \mathrm{dist}(q,p)$, and (iii) $\mathrm{dist}(p,q) + \mathrm{dist}(q,r) \geqslant \mathrm{dist}(p,r)$. These properties are called, respectively, positive-definiteness, symmetry, and the triangle inequality.

Exercise 1.1.2. (The triangle inequality) Show that $\mathrm{dist}(p,q)+\mathrm{dist}(q,r) \geqslant \mathrm{dist}(p,r)$ for all p, q in $\mathbf{R}^n$. (*Hint:* use the Cauchy-Schwartz inequality).

Finally, we define the angle between a pair of vectors in $\mathbf{R}^n$ by

$$\mathrm{angle}(p,q) := \cos^{-1}\frac{\langle p,q\rangle}{\|p\|\,\|q\|}.$$

Note that the above is well defined by the Cauchy-Schwartz inequality.

Exercise 1.1.3. (The Pythagorean theorem) Show that in a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the length of the sides (*Hint:* First prove that whenever $\langle p,q\rangle = 0$, $\|p\|^2 + \|q\|^2 = \|p-q\|^2$. Then show that this proves the theorem.).

1.2 Open sets and continuous maps

An n-dimensional open ball of radius r centered at p is defined by

$$B^n_r(p) := \{\, x \in \mathbf{R}^n \mid \mathrm{dist}(x,p) < r \,\}.$$

We say a subset $U \subset \mathbf{R}^n$ is open if for each p in U there exists an $\epsilon > 0$ such that $B^n_\epsilon(p) \subset U$. Let $A \subset \mathbf{R}^n$ be an arbitrary subset, and $U \subset A$. We say that U is open in A if there exists an open set $V \subset \mathbf{R}^n$ such that $U = A \cap V$. A mapping $f\colon A \to B$ between arbitrary subsets of $\mathbf{R}^n$ is said to be continuous if for every open set $U \subset B$, $f^{-1}(U)$ is open is A. Intuitively, we may think of a continuous map as one which sends nearby points to nearby points. A rigorous formulation of this is given in the following exercise:

Exercise 1.2.1. Let A, $B \subset \mathbf{R}^n$ be arbitrary subsets, $f\colon A \to B$ be a continuous map, and $p \in A$. Show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\operatorname{dist}(x, p) < \delta$, then $\operatorname{dist}(f(x), f(p)) < \epsilon$.

Two subset A, $B \subset \mathbf{R}^n$ are said to be homeomorphic, or topologically equivalent, if there exists a mapping $f\colon A \to B$ such that f is one-to-one, onto, continuous, and has a continuous inverse. Such a mapping is called a homeomorphism.

Definition 1.2.2. We say a subset $M \subset \mathbf{R}^3$ is an **embedded surface** if every point in M has an open neighborhood in M which is homeomorphic to an open subset of $\mathbf{R}^2$.

Note that, as an immediate consequence of the above definition, an open subset of a surface is a surface. It is not difficult to check that many of the objects which are commonly called a surface satisfy the above definition.

Exercise 1.2.3. (Stereographic projection) Show that the standard sphere $\mathbf{S}^2 := \{p \in \mathbf{R}^3 \mid \|p\| = 1\}$ is an embedded surface (*Hint*: Show that the stereographic projection π_+ form the north pole gives a homeomorphism between $\mathbf{R}^2$ and $\mathbf{S}^2 - (0,0,1)$. Similarly, the stereographic projection π_- from the south pole gives a homeomorphism between $\mathbf{R}^2$ and $\mathbf{S}^2 - (0,0,-1)$; $\pi_+(x,y,z) := (\frac{x}{1-z}, \frac{y}{1-z}, 0)$, and $\pi_-(x,y,z) := (\frac{x}{z-1}, \frac{y}{z-1}, 0)$).

Exercise 1.2.4. (Surfaces as graphs) Let $U \subset \mathbf{R}^2$ be an open subset and $f\colon U \to \mathbf{R}$ be a continuous map. Then

$$\operatorname{graph}(f) := \{(x, y, f(x,y)) \mid (x,y) \in U\}$$

is a surface. (*Hint*: Show that the orthogonal projection $\pi(x,y,z) := (x,y)$ gives the desired homeomorphism).

Note that by the above exercise the cone given by $z = \sqrt{x^2 + y^2}$, and the troughlike surface $z = |x|$ are examples of embedded surfaces. These surfaces, however, are not regular, as we will define in the next section. From the point of view of differential geometry it is desirable that a surface be without sharp corners or vertices.

1.3 Smoothness and regularity

Let $U \subset \mathbf{R}^n$ be open, and $f\colon U \to \mathbf{R}^m$ be a map. Note that f may be regarded as a list of m functions of n variables: $f(p) = (f^1(p), \dots, f^m(p))$, $f^i(p) = f^i(p^1, \dots, p^n)$. The first order partial derivatives of f are given by

$$D_j f^i(p) := \lim_{h \to 0} \frac{f^i(p^1, \dots, p^j + h, \dots, p^n) - f^i(p^1, \dots, p^j, \dots, p^n)}{h}.$$

If all the functions $D_j f^i\colon U \to \mathbf{R}$ exist and are continuous, then we say that f is differentiable (C^1). We say that f is smooth (C^∞) if the partial derivatives of f of all order exist and are continuous. These are defined by

$$Dj_1, j_2, \dots, j_k f^i := D_{j_1}(D_{j_2}(\cdots(D_{j_k} f^i)\cdots)).$$

Let $f\colon U \subset \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map, and $p \in U$. Then the Jacobian of f at p is an $m \times n$ matrix defined by

$$J_p(f) := \begin{pmatrix} D_1 f^1(p) & \cdots & D_1 f^n(p) \\ \vdots & & \vdots \\ D_m f^1(p) & \cdots & D_m f^n(p) \end{pmatrix}.$$

We say that p is a regular point of f if the rank of $J_p(f)$ is equal to n. If f is regular at all points $p \in U$, then we say that f is regular.

Exercise 1.3.1. Let $f\colon U \subset \mathbf{R}^2 \to \mathbf{R}$ be a differentiable map. Show that the mapping $X\colon U \to \mathbf{R}^3$, defined by $X(p^1, p^2) := (p^1, p^2, f(p^1, p^2))$ is a regular map.

Exercise 1.3.2. (The differential map) Let $f\colon U \subset \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map and $p \in U$. Then the differential of f at p is a mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$ defined by

$$df_p(x) := \lim_{t \to 0} \frac{f(p + tx) - f(p)}{t}.$$

Show that (i) $df_p(x) = J_p(f)\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$. Conclude then that (ii) df_p is a linear map, and (iii) p is a regular value of f if and only if df_p is one-to-one. Further, (iv) show that if f is a linear map, then $df_p(x) = f(x)$, and (v) $J_p(f)$ coincides with the matrix representation of f with respect to the standard basis.

1.4 Regular patches and surfaces

By a regular patch we mean a pair (U, X) where $U \subset \mathbf{R}^2$ is open and $X : U \to \mathbf{R}^3$ is a mapping such that (i) X is one-to-one, (ii) X is smooth, and (iii) X is regular. Furthermore, we say that the patch is proper if X^{-1} is continuous.

We are ready at last to define a regular embedded surface in $\mathbf{R}^3$:

Definition 1.4.1. We say a subset $M \subset \mathbf{R}^3$ is a **regular embedded surface**, if for each point $p \in M$ there exists a regular proper patch (U, X) and an open set $V \subset \mathbf{R}^3$ such that $X(U) = M \cap V$.

Note that X, as defined above, is a homeomorphism between U and $M \cap V$. Thus a regular surface is indeed a surface as we had defined earlier in Section 1.2. If $M \subset \mathbf{R}^3$ is a surface and (U, X) is as in the above definition, then we say that (U, X) is a local parameterization for M at p.

Exercise 1.4.2. Let $f : U \subset \mathbf{R}^2 \to \mathbf{R}$ be a smooth map. Show that $\text{graph}(f)$ is a regular embedded surface, see Exercise 1.3.1.

Exercise 1.4.3. Show that $\mathbf{S}^2$ is a regular surface (*Hint:* (Method 1) Let $p \in \mathbf{S}^2$. Then p^1, p^2, and p^3 cannot vanish simultaneously. Suppose, for instance, that $p^3 \neq 0$. Then, we may set $U := \{u \in \mathbf{R}^2 \mid \|u\| < 1\}$, and let $X(u^1, u^2) := (u^1, u^2, \pm\sqrt{1 - (u^1)^2 - (u^2)^2})$ depending on whether p^3 is positive or negative. The other cases involving p^1 and p^2 may be treated similarly. (Method 2) Write the inverse of the stereographic projection, see Exercise 1.2.3, and show that it is a regular map).

The following exercise shows that smoothness of a patch is not sufficient to ensure that the corresponding surface is without singularities (sharp edges or vertices). Thus the regularity condition imposed in Definition 1.4.1 is not superfluous.

Exercise 1.4.4. Let $M \subset \mathbf{R}^3$ be the graph of the function $f(x, y) = |x|$. Sketch this surface, and show that there exists a smooth one-to-one map $X : \mathbf{R}^2 \to \mathbf{R}^3$ such that $X(\mathbf{R}^2) = M$ (*Hint:* Let $X(x, y) := (e^{-1/x^2}, y, e^{-1/x^2})$, if $x > 0$; $X(x, y) := (-e^{-1/x^2}, y, e^{-1/x^2})$, if $X < 0$; and, $X(x, y) := (0, 0, 0)$, if $x = 0$).

The following exercise demonstrates the significance of the requirement in Definition 1.4.1 that X^{-1} be continuous.

Exercise 1.4.5. Let $U := \{(u,v) \in \mathbf{R}^2 \mid -\pi < u < \pi,\ 0 < v < 1\}$, define $X\colon U \to \mathbf{R}^3$ by $X(u,v) := (\sin(u), \sin(2u), v)$, and set $M := X(U)$. Sketch M and show that X is smooth, one-to-one, and regular, but X^{-1} is not continuous.