NONRIGIDITY OF SMOOTH TIGHT SURFACES

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Abstract. We construct smooth isometric embeddings of closed surfaces of positive genus in Euclidean 3-space which are tight, i.e., have minimal total positive curvature, but are not congruent. These embeddings agree outside a compact subset of the region of negative curvature, and enclose different volumes. Thus the rigidity theorem of Alexandrov for analytic tight surfaces does not hold in the smooth category, which settles problems of Nirenberg and Banchoff–Kühnel.

1. Introduction

A closed surface in $\mathbf{R}^3$ is *tight* if its total positive curvature is 4π [8]. These objects are natural generalizations of convex surfaces to any topological genus, and are characterized by the property that any plane cuts them into at most two pieces. Alexandrov [2] proved that analytic tight surfaces are *rigid*: they are congruent whenever they are isometric. Nirenberg [17,18,23] obtained a smooth version under additional hypotheses, including that each component of the negatively curved region contains at most one closed asymptotic curve. Han and Khuri [19] allowed several closed asymptotic curves satisfying a nondegeneracy condition. Nirenberg had conjectured [15,29] that the closed asymptotic curves do not occur, which was disproved in [5]. On the other hand Banchoff [3] constructed nonrigid tight polyhedral tori, and Banchoff and Kühnel asked whether two isometric smooth tight tori must be congruent [4]. Here we settle these problems:

Theorem 1.1. *For every integer $g \geq 1$, there is a Riemannian metric on the closed orientable surface M of genus g which admits uncountably many smooth ($\mathcal{C}^\infty$) noncongruent tight isometric embeddings into $\mathbf{R}^3$. These embeddings coincide outside a compact set in the negatively curved region of M and enclose different volumes. Furthermore 0 is a regular value of the curvature of M.*

Thus Alexandrov's rigidity theorem does not hold in the smooth category, and the answer to the Banchoff–Kühnel question is also negative. Furthermore, these surfaces satisfy the other hypothesis in Nirenberg's theorem that the gradient of the curvature does not vanish on the boundary of the negatively curved region; so his assumption on closed asymptotic curves cannot be removed. The first-named author and Spruck [16] showed that a positively curved surface with boundary is rigid if its boundary is held fixed; the above theorem shows that this fails for negative curvature. We do not know,

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however, whether these surfaces are *flexible*, i.e., admit a continuous one-parameter family of noncongruent isometric embeddings.

The proof of Theorem 1.1 is based on the following observation, which goes back to Cohn-Vossen [11]; see also Rembs [26]. Let $X\colon M \to \mathbf{R}^3$ be an embedding of a closed surface with induced metric $\mathrm{I}_X := \langle dX, dX\rangle$. A vector field Y along X is an *infinitesimal bending* [27] if $\langle dX(v), dY(v)\rangle = 0$ for all tangent vectors v. Then

$$\mathrm{I}_{X+\varepsilon Y} \;=\; \mathrm{I}_{X-\varepsilon Y} \;=\; \mathrm{I}_X + \varepsilon^2\langle dY, dY\rangle. \tag{1}$$

For small ε, the embeddings $X \pm \varepsilon Y$ therefore induce the same metric, while their enclosed volumes differ to first order by $2\varepsilon \int \langle Y, \nu\rangle$, where ν is the outward unit normal of X. Thus it suffices to construct a tight embedding $X\colon M \to \mathbf{R}^3$ carrying an infinitesimal bending of nonzero *flux* $\int_M \langle Y, \nu\rangle$ supported in the negatively curved region, for then $X \pm \varepsilon Y$ remain tight for small ε. Uncountably many embeddings are then obtained from a sequence of such bending fields with disjoint supports.

The above plan will be implemented as follows. In Section 2 we construct a negatively curved band ruled by the principal normals of a closed space curve. Near this curve the asymptotic curves transverse to the rulings are closed, which allows us to construct infinitesimal bendings of the band supported between any two of them. We then show that their flux can be made nonzero. In Section 3 we extend the band to a negatively curved annulus whose ends are surfaces of revolution. Finally, in Section 4 we double this annulus, see Figure 1, and insert g copies of it between the flat faces of a convex

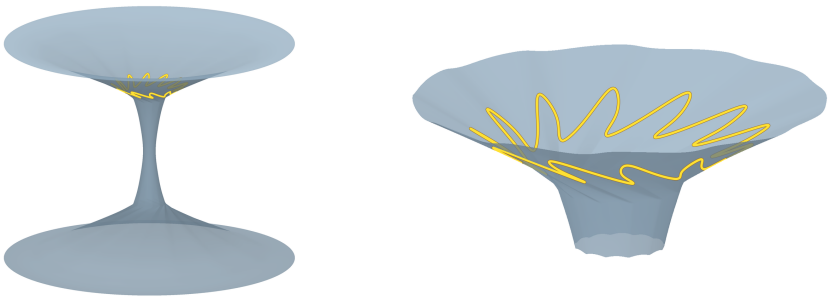


FIGURE 1. The doubled annulus (left) and an enlarged view of its upper neck (right). The band is shown in yellow.

surface, see Figure 5, which yields a tight surface of genus g. A perturbation of this surface near the flat faces then makes 0 a regular value of its curvature. The proofs are entirely analytic. The accompanying Mathematica notebook [14] illustrates the construction with the 12-fold symmetric example shown in the figures in this work.

The study of rigidity of surfaces goes back to Cauchy [1, 7] who proved it for convex polyhedra. Cohn-Vossen [10, 16] obtained the smooth version of Cauchy's theorem, and Pogorelov [24] finally established the result for all convex surfaces. On the other hand, Connelly [12] constructed examples of flexible closed polyhedral surfaces, and no smooth counterparts have yet been found. See [6] for some remarkable recent examples

of nonrigid analytic surfaces. Asymptotic curves play an important role in rigidity of nonconvex surfaces, since they form characteristics of the hyperbolic PDE governing the isometric embeddings of the negatively curved regions; see [15, 22] for other recent results on closed asymptotic curves.

2. Construction of the Band

We begin by constructing a negatively curved ruled band X which, near its central curve, is foliated by closed asymptotic curves, and show that it carries in its interior compactly supported infinitesimal bending fields Y with nonzero flux. The band is ruled by the principal normals of a closed space curve α, which is determined by its binormal indicatrix B and its speed v. Sections 2.1–2.3 assume B and v are given, and produce the band X and its bending fields Y. Section 2.4 computes the flux of Y. Section 2.5 constructs B and v, and Section 2.6 shows that X is embedded.

2.1. **The band X.** Let B be a smooth simple closed curve in the open upper hemisphere of $\mathbf{S}^2$, parametrized by arclength s with period L. Set $N := B'$, $T := -B \times N$, and write $N' = -B - \kappa T$, $T' = \kappa N$, which defines the geodesic curvature κ of B. We seek a positive L-periodic speed function v satisfying the conditions

$$\Lambda(v) := \int_0^L v\,T\,ds = 0, \qquad \Pi(v) := \int_0^L \frac{\kappa'}{\sqrt{v}}\,ds = 0. \tag{2}$$

The first condition above closes the space curve α defined below by Lelieuvre's formula [21], which was used in [5]:

$$\alpha' := vT = -vB \times B'.$$

The second condition closes the asymptotic curves on the band X we describe next. Here (T, N, B) is the Darboux frame of α with respect to X; it coincides with the Frenet frame of α up to sign wherever the curvature of α does not vanish, but is defined even at inflections [15, Sec. 2]. Define $X\colon \mathbf{R}/(L\mathbf{Z}) \times [-\delta, \delta] \to \mathbf{R}^3$ by

$$X(s, u) := \alpha(s) + u\,N(s),$$

where the *thickness* $\delta > 0$ is a small constant to be specified later in Section 2.6. Setting $w := v - \kappa u$ and $E := w^2 + u^2$, we compute that $X_s = wT - uB$, $X_u = N$, and $X_s \times X_u = wB + uT$. Hence the first fundamental form is $E\,ds \otimes ds + du \otimes du$, the area measure is $dA = \sqrt{E}\,ds\,du$, and $\nu := (wB + uT)/\sqrt{E}$ is a unit normal of X. The second fundamental form with respect to ν, multiplied by $\sqrt{E}$, is

$$u(v' - \kappa' u)\,ds \otimes ds - v\,(ds \otimes du + du \otimes ds). \tag{3}$$

In particular the curvature $K = -v^2/E^2 < 0$ as desired. Using these bands to generate closed asymptotic curves on negatively curved surfaces goes back to Kovaleva [20]. They feature also in the recent work of Ni, Zhang, and Zhang [22].

2.2. **The asymptotic curves.** A curve on a surface is *asymptotic* if its velocity is a null direction of the second fundamental form. Since X has negative curvature, precisely two asymptotic curves pass through each of its points: a ruling, and a curve transverse to the rulings. By (3), the latter curves are given by $s \mapsto X(s, u(s))$ with $u(v' - \kappa' u) - 2vu' = 0$. In particular α itself is asymptotic. For $u \neq 0$, the substitution $1/u$ makes this equation linear, and multiplying by $\sqrt{v}$ then gives

$$\left(\frac{\sqrt{v}}{u}\right)' = \frac{\kappa'}{2\sqrt{v}}. \tag{4}$$

By the second condition in (2), the right side of (4) is the derivative of a periodic function ω, which we choose so large that $\sqrt{v}/\omega < \delta$. Then the function

$$\xi(s, u) := \frac{\sqrt{v}}{u} - \omega$$

is constant along the asymptotic curves transverse to the rulings. So the level sets $\xi = c$, $c > 0$, are closed asymptotic curves, which foliate the region $0 < u < \sqrt{v}/\omega$. By contrast, in [5] the rulings were chosen first, with constant slope, and were not orthogonal to the corresponding curve. There the analogue of (4) had only one periodic solution. On the other hand, the surface was more explicit, and its rulings swept out an annulus which only had to be made tangent to planes along its boundary.

2.3. **The bending field Y.** Now we construct the desired vector fields on the band X. Each is supported between two closed asymptotic curves, and is determined by a function ϕ, which we call its *profile function*.

Proposition 2.1. *Let $I = (c_1, c_2) \subset (0, \infty)$ be a bounded open interval, and $\phi \in \mathcal{C}^\infty_c(I)$. Then the field $Y := PT + QB$, where*

$$P := -\frac{u}{2\sqrt{v}}\,\phi'(\xi), \qquad Q := u\,\phi(\xi) - \frac{w}{2\sqrt{v}}\,\phi'(\xi), \tag{5}$$

is an infinitesimal bending of X, whose support lies in the region $c_1 < \xi < c_2$ in the interior of the domain of X.

Proof. Since $X_u = N$ and $Y_u = P_u T + Q_u B$, we have $\langle X_u, Y_u \rangle = 0$. Thus Y is an infinitesimal bending if and only if $\langle X_s, Y_s \rangle = 0$ and $\langle X_s, Y_u \rangle + \langle X_u, Y_s \rangle = 0$. Substituting $X_s = wT - uB$ and $Y_s = P_s T + Q_s B + (\kappa P + Q)N$ gives

$$wP_s - uQ_s = 0, \qquad wP_u - uQ_u + \kappa P + Q = 0.$$

Writing $Q = uf + (w/u)P$, and using $w_u = -\kappa$ and $w + \kappa u = v$, the second equation becomes $P = u^3 f_u/(2v)$, and then the first equation becomes

$$f_s + \frac{u(v' - \kappa' u)}{2v}\, f_u = 0.$$

Thus f must be constant along the asymptotic curves transverse to the rulings, which are the characteristics of this equation, and conversely any such f yields a bending field via the above formulas. Since ξ is constant along these curves we may take $f := \phi(\xi)$, and then $\xi_u = -\sqrt{v}/u^2$ yields (5). Since ϕ has compact support in I, the support of Y

lies in the region $c_1 < \xi < c_2$. This region lies in the interior of the domain of X, since $\sqrt{v}/\omega < \delta$, and so Y extends smoothly by zero. □

Figure 2 shows the effect of the two signs of Y on the same patch of the band.

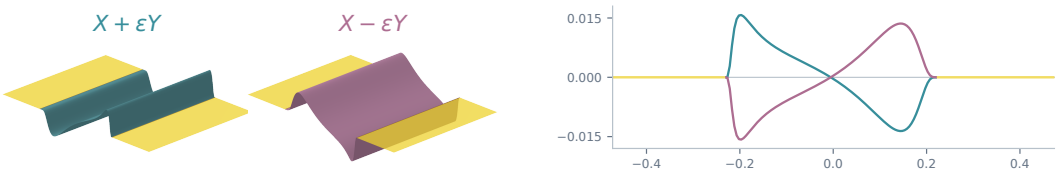


FIGURE 2. Bendings of a patch of the ruled band, with the normal coordinate enlarged eightfold. At right are the orthogonal projections of the curves $u \mapsto (X \pm \varepsilon Y)(s_0, u)$ onto the normal plane of α at $\alpha(s_0)$.

2.4. **The flux.** Next we compute the flux of the bending field Y over the band, with respect to the unit normal $\nu = (wB + uT)/\sqrt{E}$ of X from Section 2.1, and show that the profile function ϕ can be chosen to make this flux nonzero.

Lemma 2.2. *Let Y be the bending field (5) with profile $\phi \in \mathcal{C}_c^\infty(I)$. Then*

$$\int \langle Y, \nu \rangle = 2 \int_I \phi(c) \int_0^L \left(\frac{\kappa\, v^{3/2}}{(c+\omega)^4} - \frac{(1+\kappa^2)\, v}{(c+\omega)^5} \right) ds\, dc. \tag{6}$$

Proof. Since $dA = \sqrt{E}\, ds\, du$ and $\sqrt{E}\, \nu = wB + uT$, the flux density is

$$\big\langle Y,\, wB + uT \big\rangle = Pu + Qw = uw\, \phi(\xi) - \frac{E}{2\sqrt{v}}\, \phi'(\xi).$$

Since $\xi_u = -\sqrt{v}/u^2$, we may integrate the ϕ' term by parts in u. There is no boundary term, and the coefficient of $\phi(\xi)$ is

$$uw - \partial_u \left(\frac{Eu^2}{2v} \right) = \frac{2u^2}{v} (\kappa w - u).$$

Now put $c = \xi$. Substitution of $u = \sqrt{v}/(c+\omega)$ and $|du/dc| = u^2/\sqrt{v}$ gives (6). □

Proposition 2.3. *For every nonempty open interval $I \subset (0, \infty)$ there is a function $\phi \in \mathcal{C}_c^\infty(I)$ whose bending field (5) has nonzero flux.*

Proof. Suppose every $\phi \in \mathcal{C}_c^\infty(I)$ gives zero flux. Then the inner integral in (6), being continuous in c, vanishes throughout I. At any fixed $c \in I$, its jth derivative therefore vanishes for every integer $j \geq 0$, which gives

$$\int_0^L \frac{\kappa\, v^{3/2}}{(c+\omega)^{j+4}}\, ds \;=\; \frac{j+4}{4} \int_0^L \frac{(1+\kappa^2)\, v}{(c+\omega)^{j+5}}\, ds.$$

Dividing by $\int_0^L (c+\omega)^{-j-4}\, ds$ gives $(j+4) \min\big((1+\kappa^2)v/(c+\omega)\big) \leq 4 \max(\kappa v^{3/2})$. The minimum is positive, so this is impossible for large j. □

2.5. **The closing conditions.** We first choose the spherical curve B and a positive speed function v_0 satisfying the first condition in (2). Then we perturb v_0 to satisfy the second condition without losing the first. Identify $\mathbf{R}^3$ with $\mathbf{R}^2 \times \mathbf{R}$, and call the first factor the horizontal plane. Let J denote counterclockwise rotation of $\mathbf{R}^2$ through $\pi/2$, and $e_\theta := (\cos\theta, \sin\theta)$.

2.5.1. *The first condition.* We will obtain B as the radial projection into $\mathbf{S}^2$ of a closed curve $(\beta, 1)$, where β is a closed curve in $\mathbf{R}^2$; then B lies in the open upper hemisphere, as required, and is simple whenever β is; see Figure 3. We take β invariant under rotation

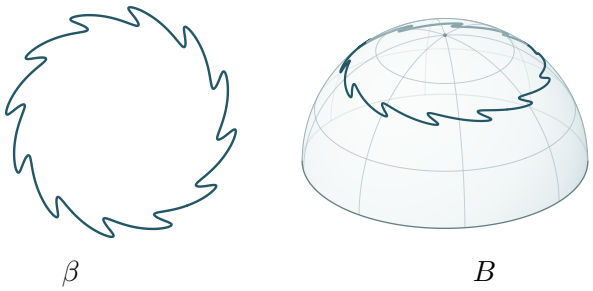


FIGURE 3. The plane curve β with frequency $q = 12$ and its spherical lift B.

through $2\pi/q$, where the *frequency* $q \geq 2$ is an integer to be chosen sufficiently large in Section 2.6, so that the horizontal part of the first condition in (2) holds automatically. The vertical part then forces $\det(\beta, d\beta/d\theta)$ to change sign, so β cannot be star-shaped about the origin. We therefore let the angle of β backtrack periodically, and add a radial wobble which separates the strands where it does, so that β remains simple. Set

$$\beta(\theta) := r(\theta) e_{\widetilde{\theta}}, \qquad r(\theta) := 1 + \frac{\sin q\theta}{q}, \qquad \widetilde{\theta} := \theta + \frac{2\sin q\theta}{q}.$$

The curve β is simple: if $\beta(\theta_1) = \beta(\theta_2)$, then $r(\theta_1) = r(\theta_2)$, so $\sin q\theta_1 = \sin q\theta_2$, and the corresponding values of $\widetilde{\theta}$ differ by $\theta_1 - \theta_2$, which is therefore a multiple of 2π. Furthermore,

$$\frac{d\beta}{d\theta} = \cos q\theta \, e_{\widetilde{\theta}} + r(1 + 2\cos q\theta) \, J e_{\widetilde{\theta}}$$

never vanishes. Thus $B = (\beta, 1)/\sqrt{1 + r^2}$ is a smooth simple closed curve, as required in Section 2.1. Increasing θ by $2\pi/q$ rotates β through $2\pi/q$ about the origin, and B through $2\pi/q$ about the z-axis. We regard θ as a function of the arclength s of B; since rotations preserve arclength, increasing s by L/q increases θ by $2\pi/q$. Since $T = (dB/ds) \times B$, direct calculation gives

$$T\, ds = \frac{1}{1 + r^2} \left(-J\frac{d\beta}{d\theta}, -\det\left(\beta, \frac{d\beta}{d\theta}\right)\right) d\theta.$$

For any L/q-periodic speed v, the horizontal part of $\int_0^L vT\,ds$ vanishes by rotational symmetry. To make its vertical part vanish as well, set

$$\lambda(t) \coloneqq 1-\cos t+\frac{1}{4}\cos 2t=\frac{1}{4}+\frac{1}{2}(1-\cos t)^2, \qquad v_0(\theta)\coloneqq \frac{1+r(\theta)^2}{r(\theta)^2}\,\lambda(q\theta).$$

Since $\det(\beta, d\beta/d\theta)=r^2(1+2\cos q\theta)$, the vertical component of $\int_0^L v_0T\,ds$ is

$$-\int_0^{2\pi}\lambda(q\theta)(1+2\cos q\theta)\,d\theta=0,$$

because the means of $\lambda(t)$ and $\lambda(t)\cos t$ are 1 and $-1/2$, respectively. Thus $v_0>0$ satisfies the first condition in (2).

2.5.2. *The second condition.* We now decrease the speed v_0 on a short interval where κ' has the sign opposite to that of $\Pi(v_0)$, which makes Π change sign, while a compensating perturbation elsewhere preserves the first condition. Set $\Lambda_3(v)\coloneqq\langle\Lambda(v),e_3\rangle$. For L/q-periodic v the horizontal components of $\Lambda(v)$ vanish by symmetry, as in Section 2.5.1, so the first condition $\Lambda(v)=0$ reduces to $\Lambda_3(v)=0$. If $\Pi(v_0)=0$, take $v=v_0$. Otherwise κ is not constant, and since it is L/q-periodic, κ' takes both signs on a period. Choose a short interval on which κ' has sign opposite to $\Pi(v_0)$, and a smooth cutoff $0\le\chi\le 1$ supported there and equal to one on a smaller interval. Since $\langle T,e_3\rangle$ is not identically zero, we may also choose a fixed bump ψ away from the interval with $\Lambda_3(\psi)\neq 0$. Extend both cutoffs L/q-periodically and set

$$v_\sigma\coloneqq v_0(1-\sigma\chi)+\sigma\frac{\Lambda_3(v_0\chi)}{\Lambda_3(\psi)}\,\psi, \qquad 0\le\sigma<1.$$

The last term ensures that $\Lambda_3(v_\sigma)=0$. By shrinking the support of χ, we make its coefficient arbitrarily small. Since the supports are disjoint, the speeds remain positive and we may arrange that $\int_0^L|v_\sigma-v_0|\,ds<1/q$ for all σ. Where $\chi=1$, the contribution to $\Pi(v_\sigma)$ diverges as $\sigma\to 1$ with sign opposite to $\Pi(v_0)$. The rest of the support of χ contributes with the same sign, and the integral outside that support stays bounded. Hence $\Pi(v_\sigma)=0$ for some σ. Taking $v=v_\sigma$ gives both conditions in (2). The curve B and speed v now define the band X of Section 2.1.

2.6. **Embeddedness.** Finally we show that X is embedded when the frequency q is sufficiently large and the thickness δ is sufficiently small. By compactness, it suffices to check that α is embedded. Let $\overline{\alpha}$ be the projection of α into the horizontal plane, and translate α so that $\overline{\alpha}$ has mean zero with respect to θ. We claim that, uniformly in θ,

$$\beta(\theta)=e_\theta+O(1/q), \qquad \overline{\alpha}(\theta)=\tfrac{1}{2}e_\theta+O(1/q).$$

The first estimate follows from the definitions of r and $\widetilde{\theta}$. For the second, let $\overline{\alpha}_0$ be the projection of the curve with speed v_0. By the formula for $T\,ds$,

$$\frac{d\overline{\alpha}_0}{d\theta}=-\frac{\lambda(q\theta)}{r^2}\,J\frac{d\beta}{d\theta}=\lambda(q\theta)\big((1+2\cos q\theta)e_\theta-\cos q\theta\,Je_\theta\big)+O(1/q).$$

The coefficients of e_θ and Je_θ on the right have means zero and $1/2$, respectively. A periodic function of mean zero evaluated at $q\theta$ has an antiderivative which is $O(1/q)$.

Thus, by integration by parts, its integral against e_θ or Je_θ on any subinterval of $[0, 2\pi]$ is $O(1/q)$. Since $de_\theta/d\theta = Je_\theta$, it follows that $\overline{\alpha}_0 = \frac{1}{2}e_\theta + O(1/q)$ up to translation. Furthermore, if $\overline{\alpha}$ and $\overline{\alpha}_0$ start at the same point, then $|\overline{\alpha} - \overline{\alpha}_0| \leq \int_0^L |v - v_0|\, ds < 1/q$. The choice of mean zero now gives the claim. In particular, $\overline{\alpha}$ winds once around the origin. To see that $\overline{\alpha}$ is simple, note that $d\overline{\alpha}/d\theta$ is a positive multiple of $-J\, d\beta/d\theta$, and

$$\Big\langle -J\frac{d\beta}{d\theta},\, e_{\theta+\pi/3}\Big\rangle = \frac{1}{2}\Big(1 + \big(2 - \sqrt{3}\,\big)\cos q\theta\Big) + O(1/q).$$

This is bounded below by a positive constant. If $\overline{\alpha}(\theta_1) = \overline{\alpha}(\theta_2)$ for $0 < \theta_2 - \theta_1 \leq \pi$, the uniform estimate for $\overline{\alpha}$ gives $\theta_2 - \theta_1 = O(1/q)$. Since $d\beta/d\theta$ is uniformly bounded, the displayed inequality remains strict on this interval with the fixed vector $e_{\theta_1+\pi/3}$ in place of $e_{\theta+\pi/3}$. Thus $\langle \overline{\alpha}, e_{\theta_1+\pi/3}\rangle$ is strictly increasing between θ_1 and θ_2, a contradiction. Thus $\overline{\alpha}$, and consequently α, is embedded. So X is embedded for δ sufficiently small.

3. Extension of the Band

Here we extend the band X to an annulus with rotationally symmetric ends (this surface forms the bottom half of the annulus depicted in Figure 1). We work with the support function h of X, a function on an annular domain whose gradient maps the domain onto the horizontal projection of X. This is the classical device, going back to Pogorelov and Cheng–Yau, which links the Minkowski problem to the Monge–Ampère equation [9, 25, 28]. In these terms negative curvature means $\det \operatorname{Hess} h < 0$, and rotational symmetry means that h is radial.

3.1. **The support function and its Legendre transform.** Choosing the thickness δ of X sufficiently small, we may assume that the Gauss map ν of X is one-to-one into the open upper hemisphere, and the horizontal projection of X is an embedding; indeed, along α these maps restrict to the embedded curves B and $\overline{\alpha}$, and are local diffeomorphisms, since $K < 0$ and B lies in the open upper hemisphere.

The *normal coordinate* of a point p of X is the point $x(p) \in \mathbf{R}^2$ which is determined by $\nu(p) = (x(p), 1)/\sqrt{1 + |x(p)|^2}$. Thus $(x(p), 1)$ is the point where the ray from the origin through $\nu(p)$ meets the plane $z = 1$. In particular the normal coordinate of α is β, by Section 2.5.1. Conversely, since ν is one-to-one, x determines p, and we let

$$h(x) \coloneqq \langle p(x), (x, 1)\rangle$$

be the *support function* of X. Since $(x, 1)$ is normal to X at $p(x)$, differentiation shows that $\nabla h(x)$ is the horizontal projection of $p(x)$. So X is parametrized by

$$x \longmapsto \big(\nabla h,\ h - \langle x, \nabla h\rangle\big), \tag{7}$$

with curvature $K = 1/\big((1 + |x|^2)^2 \det \operatorname{Hess} h\big)$. In particular $\nabla h(\beta) = \overline{\alpha}$, ∇h is one-to-one, and $\det \operatorname{Hess} h < 0$. We say that a smooth function h on a domain in $\mathbf{R}^2$ is *saddle* if $\det \operatorname{Hess} h < 0$. Then, conversely, (7) is a negatively curved immersion with normal $(x, 1)$. It is an embedding if ∇h is one-to-one, in which case we let

$$h^*(y) \coloneqq \langle x, y\rangle - h(x)$$

be the *Legendre transform* of h, where x is determined by $\nabla h(x) = y$. Then $\nabla h^*(y) = x$ and $\operatorname{Hess} h^* = (\operatorname{Hess} h)^{-1}$; in particular $\det \operatorname{Hess} h^* < 0$. Furthermore, by (7), the surface with support function h is the graph of $-h^*$.

3.2. **The extension problem.** Recall from Proposition 2.1 that the bending field Y is supported in the region $c_1 < \xi < c_2$ of X. By Proposition 2.3, we may assume that Y has nonzero flux. Let Ω be the domain of h, and Ω^* be the horizontal projection of X. Then $\nabla h\colon \Omega \to \Omega^*$ is a diffeomorphism, and X is the graph of $-h^*$ over Ω^*. Denote the normal coordinates of the boundary curves of X, in order from the inside, by $\gamma_1(s)$ and $\gamma_2(s)$. Then Ω is the annulus bounded by γ_1 and γ_2, and Ω^* is the annulus bounded by the curves $\gamma_i^* := \nabla h(\gamma_i)$. These four curves wind counterclockwise about the origin, while ∇h reverses orientation, since $\det \operatorname{Hess} h < 0$. So γ_2^* is the inner boundary of Ω^*. To establish the desired extension of X it suffices to show:

Proposition 3.1. *After a vertical translation of X, and a modification in an arbitrarily small collar of $\partial\Omega$, the support function h of X extends to a saddle function on $\mathbf{R}^2 \setminus \{0\}$, so that h is radial near the origin, and $h(x) = a|x| - b/|x|$ near infinity, for a, $b > 0$. Moreover ∇h is a diffeomorphism of $\mathbf{R}^2 \setminus \{0\}$ onto an annulus $a < |y| < R_0$, and h^* extends smoothly to a constant function across the circle $|y| = R_0$.*

The extended h is the support function of an embedded negatively curved annulus A, which agrees with X on the bending region $c_1 \le \xi \le c_2$, and is the graph of $-h^*$ over the annulus $a < |y| < R_0$. The outer end of A agrees to infinite order with a horizontal plane, and the inner end is the surface of revolution $|y| = a + z^2/(4b)$. The proof is given in Section 3.5. It rests on two lemmas for extending a saddle function inward: across a circle, by a radial function (Section 3.3), and across a closed curve up to a circle, by a function which is affine along line segments (Section 3.4). Applied to h, these lemmas extend it across the inner boundary γ_1 of Ω; applied to h^*, they extend h across the outer boundary γ_2. We need the following fact.

Lemma 3.2 ([13, Lem. 1.1])**.** *Let f be a local diffeomorphism from a compact connected surface with boundary into $\mathbf{R}^2$, which is one-to-one on each boundary component. Then f is an embedding.*

3.3. **Radial extension.** Here we show that a saddle function extends radially inward across a circle, provided that its Hessian is positive in the direction tangent to the circle (Lemma 3.4); the corresponding end of the surface is then a surface of revolution. First we show how such functions may be joined.

Lemma 3.3. *Let h_1 and h_2 be saddle functions on the two sides of a smooth closed curve Γ in $\mathbf{R}^2$, which agree to first order along Γ. Then within any prescribed collar U of Γ they can be joined into a single saddle function, which agrees with h_1 and h_2 outside U.*

Proof. Let f be the function equal to h_1 and h_2 on the two sides of Γ, which is $\mathcal{C}^1$ since h_1 and h_2 agree to first order. In a tangent–normal frame along Γ the first columns of the one-sided Hessians agree, by differentiation of the common gradient along Γ. Thus the determinant is affine on the segment between the two Hessians and is strictly

negative throughout, uniformly along the compact curve Γ. Let f_ε be the mollification of f with radius ε. Since f is $\mathcal{C}^1$, its distributional Hessian has no singular part along Γ, so near Γ the Hessian of f_ε is uniformly close to that segment, while away from Γ we have $f_\varepsilon \to f$ in $\mathcal{C}^2$. Fix a cutoff function χ supported in U and equal to one near Γ, and put $\widetilde{f} \coloneqq f + \chi(f_\varepsilon - f)$. Then $\widetilde{f} = f_\varepsilon$ near Γ, $\widetilde{f} = f$ outside U, and on the support of $\nabla\chi$, which lies at a positive distance from Γ, $\widetilde{f} \to f$ in $\mathcal{C}^2$. Hence $\det \operatorname{Hess} \widetilde{f} < 0$ for small ε. □

We write $\operatorname{Hess} f > 0$ *along the circle* $|y| = R$ if $\operatorname{Hess} f(v, v) > 0$ for all nonzero vectors v tangent to the circle, or equivalently, $f_{\theta\theta} + Rf_\rho > 0$ on the circle in polar coordinates.

Lemma 3.4. *Let f be a saddle function on a neighborhood of the circle $|y| = R$ in $\{|y| \geq R\}$, with $\operatorname{Hess} f > 0$ along the circle, and g be a smooth function on $(a, R]$, $0 \leq a < R$, with $g'(R) = 0$ and $g'' < 0$. Then, after a modification in an arbitrarily small collar of the circle, f extends to a saddle function on $a < |y| \leq R$, which coincides with $\mu g(|y|)$ near $|y| = a$, up to an additive constant, for a large constant μ.*

Proof. We may assume that $g(R) = 0$. Choose a cutoff χ on $(a, R]$ with $\chi = 0$ near a and $\chi = 1$ near R, and put

$$f_\mu \coloneqq \mu\, g(\rho) + \chi(\rho)\big(f(Re_\theta) + (\rho - R)\, f_\rho(Re_\theta)\big),$$

which agrees with f to first order along the circle. The radial and tangential entries of the Hessian of f_μ are $\partial_\rho^2 f_\mu$ and $(\partial_\theta^2 f_\mu + \rho\, \partial_\rho f_\mu)/\rho^2$. For large μ the first is negative and the second is positive: the terms multiplying μ are $g'' < 0$ and g'/ρ, which is positive for $\rho < R$, while the remaining terms are bounded, vanish near a, and in the second entry equal the positive function $(f_{\theta\theta} + Rf_\rho)/R^2$ at $\rho = R$. So $\det \operatorname{Hess} f_\mu < 0$ regardless of the mixed entry, and Lemma 3.3 joins f and f_μ in a collar of the circle. □

3.4. **Extension to a circle.** Here we extend a saddle function f inward, across a closed curve γ enclosing the origin up to a circle centered at the origin, so that $\operatorname{Hess} f > 0$ along that circle (Lemma 3.5). The extension is affine along line segments which join γ to the circle, so its graph is a ruled surface. We need γ and its image γ^* under ∇f to satisfy the following condition.

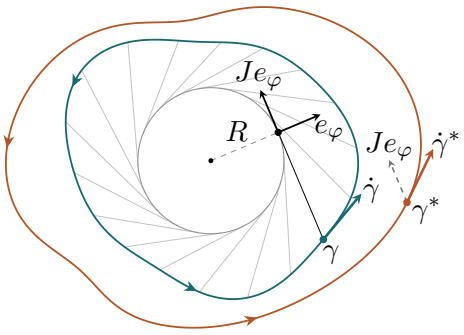


FIGURE 4. A pair of curves (γ, γ^*) satisfying the circle condition at R: $\dot\gamma$ has a positive component along e_φ, and $\dot\gamma^*$ along Je_φ.

For every point $y \in \mathbf{R}^2$ with $|y| > R > 0$ there is a unique angle $\varphi(y)$, mod 2π, such that $y = Re_\varphi - \ell\, Je_\varphi$, $\ell > 0$. So y lies on the line tangent to the circle of radius R about the origin at Re_φ, and the segment from y to Re_φ points in the counterclockwise direction Je_φ; see Figure 4. We say that a pair (γ, γ^*) of regular closed curves in $\mathbf{R}^2$, with a common parameter s, satisfies the *circle condition* at R if $|\gamma| > R$ and,

$$\big\langle \dot\gamma(s), e_{\varphi(\gamma(s))} \big\rangle > 0, \qquad \big\langle \dot\gamma^*(s), Je_{\varphi(\gamma(s))} \big\rangle > 0.$$

The first inequality means that γ crosses the tangent segments to the circle transversely, with the point of tangency Re_φ moving counterclockwise. The second means that $\dot\gamma^*(s)$ makes an acute angle with the segment from $\gamma(s)$ to its point of tangency Re_φ.

Lemma 3.5. *Let γ be a positively oriented simple closed curve in $\mathbf{R}^2$ enclosing the origin, f be a saddle function on a neighborhood of γ in the closure of its exterior, and $\gamma^* := \nabla f(\gamma)$. Suppose that (γ, γ^*) satisfies the circle condition at R. Then, after a modification in an arbitrarily small collar of γ, f extends to a saddle function on the region between γ and the circle $|y| = R$, so that* Hess $f > 0$ *along this circle.*

Proof. Put $\varphi := \varphi(\gamma)$. Then $\gamma = Re_\varphi - \ell\, Je_\varphi$, where $\ell := \sqrt{|\gamma|^2 - R^2}$, and differentiation yields $\dot\varphi = \langle \dot\gamma, e_\varphi\rangle/\ell > 0$. Since γ winds once about the origin, φ increases by 2π. For a small constant $\tau > 0$ set $\theta := \varphi - \tau$ and $W := Re_\theta - \gamma$. The quantities

$$D_1 := \langle \dot\gamma^*, W\rangle, \qquad D_2 := \det(\dot\gamma, W), \qquad D_3 := \det\big(R\,\dot\theta\, Je_\theta, W\big) = R\,\dot\theta\,\big(\langle \gamma, e_\theta\rangle - R\big)$$

are positive for sufficiently small τ: since $W \to \ell Je_\varphi$ as $\tau \to 0$, the first two tend to $\ell\langle \dot\gamma^*, Je_\varphi\rangle$ and $\ell\langle \dot\gamma, e_\varphi\rangle$, while $\langle \gamma, e_\theta\rangle - R = \ell \sin\tau - R(1 - \cos\tau) > 0$. The map $(s,t) \mapsto \gamma + tW$, $0 \le t \le 1$, has Jacobian $\Delta := (1-t)D_2 + tD_3 > 0$, and is one-to-one on the boundary curves γ and Re_θ. So, by Lemma 3.2, it is a diffeomorphism onto the region between γ and the circle $|y| = R$. On this region let $\widehat f$ be the function which is affine on each segment $t \mapsto \gamma + tW$, and agrees with f to first order at γ, i.e.,

$$\widehat f(\gamma + tW) := f(\gamma) + t\langle \gamma^*, W\rangle.$$

Differentiating with respect to s and t yields $\nabla \widehat f(\gamma + tW) = \gamma^* - \frac{tD_1}{\Delta}\, JW$. In particular $\widehat f$ agrees with f to first order along γ. Since $\widehat f$ is affine along the segments, W is a null direction of Hess $\widehat f$. So det Hess $\widehat f \le 0$, with equality only where $(\text{Hess}\, \widehat f)W = \partial_t \nabla \widehat f$ vanishes; but $\partial_t \nabla \widehat f = -(D_1 D_2/\Delta^2)\, JW \ne 0$.

It remains to check that Hess $\widehat f > 0$ along the circle. There $\widehat f(Re_\theta) = f(\gamma) + \langle \gamma^*, W\rangle$, so $\widehat f_{\theta\theta}$ is bounded uniformly as $\tau \to 0$, since $\dot\varphi > 0$. On the other hand, by the above formula for $\nabla \widehat f$ with $t = 1$, $\widehat f_\rho = \langle \gamma^*, e_\theta\rangle + (D_1/D_3)\langle W, Je_\theta\rangle$. As $\tau \to 0$, $D_3 \to 0$ while D_1 and $\langle W, Je_\theta\rangle$ tend to the positive quantities $\ell\langle \dot\gamma^*, Je_\varphi\rangle$ and ℓ. So $\widehat f_\rho \to \infty$ uniformly, and $\widehat f_{\theta\theta} + R\widehat f_\rho > 0$ for small τ. Finally, Lemma 3.3 joins f and $\widehat f$ in a collar of γ. □

3.5. **Proof of Proposition 3.1.** First we check that the boundary curves of Ω^* and Ω satisfy the circle condition, the latter after a reflection. Let $\mathcal{R}$ denote reflection in the first coordinate axis. For a curve $\gamma(s)$ we also write $\mathcal{R}\gamma$ for its reflection with the parameter reversed, i.e., the curve $s \mapsto \mathcal{R}(\gamma(-s))$; so $\mathcal{R}\gamma$ winds counterclockwise whenever γ does.

Lemma 3.6. *If the frequency q of β is sufficiently large, and the thickness δ of X is sufficiently small, then (γ_2^*, γ_2) and $(\mathcal{R}\gamma_1, \mathcal{R}\gamma_1^*)$ satisfy the circle condition.*

Proof. As $\delta \to 0$, γ_i converge to β in $\mathcal{C}^1$, and so $\gamma_i^* = \nabla h(\gamma_i)$ converge to $\nabla h(\beta) = \overline{\alpha}$. So it suffices to check the conditions for the pairs $(\overline{\alpha}, \beta)$ and $(\mathcal{R}\beta, \mathcal{R}\overline{\alpha})$, for which, by Section 2.6, $\beta = e_\theta + O(1/q)$ and $\overline{\alpha} = \frac{1}{2}e_\theta + O(1/q)$ uniformly, and $d\overline{\alpha}/d\theta$ is a positive multiple of $-J\, d\beta/d\theta$. In both pairs (γ, γ^*), $\dot{\gamma}^*$ is therefore a positive multiple of $J\dot{\gamma}$, so the two inequalities of the circle condition reduce to $\langle \dot{\gamma}, e_\varphi \rangle > 0$. For the first pair, with $R = 1/4$, $e_\varphi = e_{\theta+\pi/3} + O(1/q)$, and $\langle \dot{\gamma}, e_\varphi \rangle$ is a positive multiple of $\langle -J\, d\beta/d\theta,\, e_{\theta+\pi/3} \rangle$, which is bounded below by a positive constant by Section 2.6. For the second pair, $\gamma = \mathcal{R}\beta = e_\theta + O(1/q)$, so, with $R = 4/5$, $e_\varphi = \frac{4}{5}e_\theta + \frac{3}{5}Je_\theta + O(1/q)$, while

$$\dot{\gamma} = -\mathcal{R}\,(d\beta/d\theta)(-\theta) = -\cos q\theta\, e_\theta + (1 + 2\cos q\theta)\, Je_\theta + O(1/q),$$

and hence $\langle \dot{\gamma}, e_\varphi \rangle = (3 + 2\cos q\theta)/5 + O(1/q) > 0$. The radius inequalities follow from the same approximations. □

Now we prove Proposition 3.1, by extending h first inward, across the inner boundary γ_1 of Ω, and then outward, across the outer boundary γ_2.

3.5.1. *Inward extension.* By Lemma 3.6, the pair $(\mathcal{R}\gamma_1, \mathcal{R}\gamma_1^*)$ of reflected curves satisfies the circle condition at some R. Here $\mathcal{R}\gamma_1$ is the inner boundary of the domain $\mathcal{R}\Omega$ of the saddle function $h(\mathcal{R}x)$, and is mapped by the gradient of this function onto $\mathcal{R}\gamma_1^*$. So Lemma 3.5 extends $h(\mathcal{R}x)$, after a modification in a small collar of $\mathcal{R}\gamma_1$, to a saddle function on the region between $\mathcal{R}\gamma_1$ and the circle $|x| = R$, whose Hessian is positive along this circle. This is the hypothesis of Lemma 3.4, which, with $a = 0$, extends the function further to the region $0 < |x| \leq R$, once we choose a function g on $(0, R]$ with $g'(R) = 0$ and $g'' < 0$. Let $\psi\colon [0,1] \to [0, R]$ be a smooth function with $\psi(0) = R$, $\psi(1) = 0$, and $\psi' < 0$ on $[0, 1)$, which is flat at 1, and g be the antiderivative of ψ^{-1} on $(0, R]$ with $g(0) = 0$. Reflecting back, we then obtain an extension of h to a saddle function on the region $|x| > 0$ inside γ_2, so that $h(x) - \mu g(|x|)$ is constant near the origin, for a large constant μ. Set $R_0 := \mu$.

3.5.2. *Outward extension.* As ∇h maps the outer boundary γ_2 of Ω onto the inner boundary γ_2^* of Ω^*, we extend h^* inward across γ_2^*, and then take the Legendre transform. The function h^* is saddle on Ω^*, and its gradient maps γ_2^* onto γ_2. By Lemma 3.6, the pair (γ_2^*, γ_2) satisfies the circle condition at some R. So Lemma 3.5 extends h^*, after a modification in a small collar of γ_2^*, to a saddle function on the region between γ_2^* and the circle $|y| = R$, whose Hessian is positive along this circle.

This is again the hypothesis of Lemma 3.4, which we apply with $0 < a < \min\{R, R_0\}$, and a function g on $(a, R]$ which equals $2\sqrt{\rho - a}$ near a, and satisfies $g'(R) = 0$ and $g'' < 0$. This extends h^* to a saddle function on the region $|y| > a$ inside γ_1^*, so that $h^*(y) = 2\sqrt{b(|y| - a)} - c$ near $|y| = a$, for some constants $b > 0$ and c. Since a vertical translation of X changes h only by an additive constant, we may assume that $c = 0$. Since the above modifications take place in small collars of γ_1 and γ_2^*, they do not affect h near β, or near the region $c_1 \leq \xi \leq c_2$.

On the annulus between $\overline{\alpha}$ and a circle $|y| = \rho$, with ρ close to a, the gradient of h^* is a local diffeomorphism which maps the boundary curves one-to-one onto β and

the circle $|x| = \sqrt{b/(\rho - a)}$. So it is an embedding by Lemma 3.2. Letting $\rho \to a$, we see that ∇h^* maps the region $|y| > a$ inside $\overline{\alpha}$ diffeomorphically onto the exterior of β. Hence the Legendre transform of h^* on this region is a saddle function on the exterior of β, which coincides with h near β, and equals $a|x| - b/|x|$ near infinity. Together with the inward extension of h, this yields the desired extension of h to $\mathbf{R}^2 \setminus \{0\}$.

Finally, on sufficiently small and large circles $|x| = \rho$, ∇h equals $\mu g'(\rho) e_\theta$ and $(a + b/\rho^2) e_\theta$ respectively, and $\mu g'(\rho) \to R_0$ as $\rho \to 0$. So Lemma 3.2, applied to the annuli between these circles, shows that ∇h is a diffeomorphism onto $a < |y| < R_0$. Furthermore, for $|y|$ close to R_0, the radial derivative of h^* is $|x| = \psi(|y|/R_0)$, and h^* converges to a constant as $|y| \to R_0$. Since ψ is flat at 1, it follows that h^* extends smoothly by this constant across the circle $|y| = R_0$. □

4. Proof of Theorem 1.1

We first assemble a tight surface Σ of genus g from copies of the annulus A, and perturb it so that 0 becomes a regular value of its curvature. Then we bend Σ along the fields Y to obtain the desired embeddings.

4.1. **The initial surface.** Let A be the annulus constructed in Section 3.2. Recall that A is the graph of $-h^*$ which is saddle, and so has no local extrema. Thus A lies between the horizontal planes of its boundaries. Since the top end of A is the surface of revolution $|y| = a + z^2/(4b)$, it extends smoothly by reflection in the plane $z = 0$. Thus A and its reflection form a smooth embedded annulus $\widetilde{A}$ with negative curvature in its interior, which agrees to infinite order with the planes of its boundary circles. Let C be a smooth closed convex surface, symmetric with respect to a horizontal plane, with flat regions on top and bottom as depicted in Figure 5. We may assume that the flat regions are large enough to contain g disjoint disks of radius equal to that of the boundaries of $\widetilde{A}$, and that the flat regions are separated by the same distance as the height of $\widetilde{A}$. Removing the disks and gluing in g copies of $\widetilde{A}$ yields a smooth closed surface Σ of genus g which is tight, since by construction its total positive curvature agrees with that of C which is 4π.

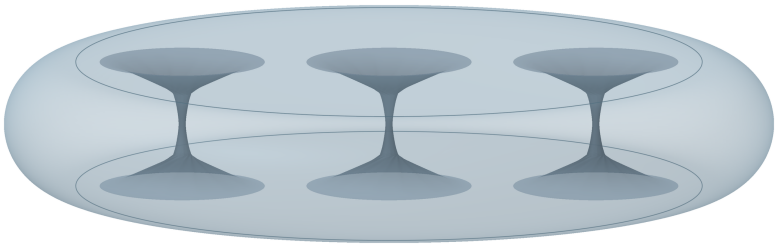


Figure 5. A genus 3 tight surface constructed using the annulus in Figure 1.

4.2. **The perturbation.** Next we perturb Σ near its flat regions, so that 0 becomes a regular value of its curvature. Let $z = H$ be the plane of the top flat region of Σ, and (y_i, H) be the centers of the upper boundary circles of the copies of $\widetilde{A}$ in Σ. Near the i-th circle, Σ is the graph $z = H + f(r)$, where $r := |y - y_i|$. By the last paragraph of Section 3.5, $f' > 0$ and $f'' < 0$ for $r < R_0$, while $f = 0$ for $r \geq R_0$.

Now, for a small constant $\lambda > 0$, choose C to be positively curved, with its top region given by the graph $z = H - \lambda|y|^2/2$ instead of being flat, and subtract $\lambda\eta(r)|y|^2/2$ from the height of the i-th copy of $\widetilde{A}$, where η is a smooth function on $(0, R_0]$ which is equal to 1 near R_0, and vanishes outside a slightly larger neighborhood of R_0. Modify the bottom of Σ symmetrically. Where η is not constant, the curvature of $\widetilde{A}$ is negative and bounded away from zero; so it remains negative for small λ. Furthermore, the modified part of $\widetilde{A}$ is disjoint from the band X.

Near the i-th circle, the new surface is the graph $z = H + f(r) - \lambda|y|^2/2$, whose Hessian has eigenvalues $f'' - \lambda < 0$ and $f'/r - \lambda$. Since f'/r decreases strictly to zero as $r \to R_0$, there is a unique radius r_λ with $f'(r_\lambda) = \lambda r_\lambda$, which is a simple zero of $f'/r - \lambda$. So the curvature is negative for $r < r_\lambda$, positive for $r > r_\lambda$, and its gradient does not vanish on the curve $r = r_\lambda$. Along this curve the gradient of the height function is $-\lambda y_i$. So the curve lies in the plane

$$z = H + f(r_\lambda) - \frac{\lambda}{2} r_\lambda^2 + \frac{\lambda}{2}|y_i|^2 - \lambda\langle y_i, y\rangle,$$

which is tangent to the surface along the curve. These planes have normals $(\lambda y_i, 1)$, and so are pairwise nonparallel. Since the horizontal projection of the curve is a circle, the curve is an ellipse, whose geodesic curvature, with respect to the conormal which points into the negatively curved region, coincides with its curvature in this plane. So each component of the negatively curved region is an annulus whose boundary curves each have total geodesic curvature 2π, and therefore has total curvature -4π by the Gauss–Bonnet theorem. Since C is positively curved, the curvature is positive elsewhere; so 0 is a regular value of the curvature. It follows that the total positive curvature of the new surface is $4\pi(1 - g) + 4\pi g = 4\pi$. So the new surface, which we again denote by Σ, is tight.

4.3. **The bendings.** Let $X\colon M \to \mathbf{R}^3$ be the embedding with image Σ constructed above. On one of the copies of $\widetilde{A}$ take the bending field Y of nonzero flux chosen in Section 3.2. Its support is compact in the region where A agrees with the band of Section 2, so it extends by zero to a smooth infinitesimal bending of X, with flux $\mathcal{F} := \int_M \langle Y, \nu\rangle \neq 0$. Since M is compact, its embeddings are open in the $\mathcal{C}^1$-topology. Thus $X_\pm := X \pm \varepsilon Y$ are embeddings for small $\varepsilon > 0$, and by (1) they induce the same metric on M. Since $\operatorname{supp} Y$ is compact in the region of negative curvature, $\mathcal{C}^2$ closeness keeps the curvature of $X_\pm$ negative there. Hence the total positive curvature remains at 4π and tightness is preserved. By the first variation of volume, the volumes $V(X_\pm)$ enclosed by $X_\pm$ satisfy $V(X_+) - V(X_-) = 2\varepsilon\mathcal{F} + o(\varepsilon)$. Since $\mathcal{F} \neq 0$, these volumes differ for small $\varepsilon > 0$, and hence $X_\pm$ are noncongruent.

To obtain uncountably many embeddings, apply Proposition 2.3 to a sequence of disjoint subintervals of I. This gives bending fields $Y_1, Y_2, \ldots$ of X with disjoint supports

and nonzero flux. Then $2\delta_j \coloneqq V(X + t_jY_j) - V(X - t_jY_j)$ is nonzero for small $t_j > 0$, as shown above. Choose t_j successively so small that $\delta_j \neq 0$, $|\delta_j| < |\delta_{j-1}|/4$ for $j \geq 2$, and $|t_jY_j|_{\mathcal{C}^{j+2}} \leq \varepsilon 2^{-j}$, where $\varepsilon > 0$ is a small constant. For $\sigma \in \{-1, 1\}^{\mathbf{N}}$ set

$$X_\sigma \coloneqq X + \sum \sigma_j t_j Y_j.$$

This series converges in every $\mathcal{C}^k$ norm, and $|X_\sigma - X|_{\mathcal{C}^2} \leq \varepsilon$. So X_σ is a smooth embedding, which is tight as above, and 0 is a regular value of its curvature, since $X_\sigma = X$ outside a compact set where the curvature is negative. Furthermore, since at each point of M at most one dY_j is nonzero, (1) shows that X_σ induces the metric $\mathrm{I}_X + \sum_j t_j^2 \langle dY_j, dY_j \rangle$, which does not depend on σ. It remains to check that the embeddings X_σ enclose different volumes. Since the supports of Y_j are disjoint, and $V(X) = \frac{1}{3}\int_M \langle X, \nu \rangle$ by the divergence theorem, with the same formula for each X_σ, the contributions of Y_j to the enclosed volume add. So, first for finite sums and then in the limit by $\mathcal{C}^1$ convergence, $V(X_\sigma) - V(X_{\sigma'}) = \sum_j (\sigma_j - \sigma'_j)\delta_j$. If $\sigma \neq \sigma'$, and j is the first index with $\sigma_j \neq \sigma'_j$, then the absolute value of this sum is at least $2|\delta_j| - 2\sum_{k>j} |\delta_k| > 0$. □

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